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REFLECTIONS ON USING CONCEPT MAPS IN TEACHING MATHEMATICS

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Abstract. The construction of concept maps in mathematics courses provides students with a rich learning experience and yields substantial insights into the degree of connectedness of their knowledge with respect to given topics. While students are given the opportunity to express their knowledge in a manner that makes sense to them, an analysis of the concept maps provides the teacher with valuable information with respect to individual and whole-class misconceptions and to the effectiveness of instruction. This paper presents a brief overview of how concept maps can be used in a variety of mathematics courses and reflections on their use in a geometry course as a means of categorizing students' geometric concept development in terms of the van Hiele levels.

1 Introduction

The development of a knowledge-base in which mathematical terms and topics are viewed as an integrated whole rather than as isolated pieces of information is strongly emphasized as a key goal in the current mathematics education literature (Bishop, Clements, Keitel, Kilpatrick, & Laborde, 1996; Hiebert and Carpenter, 1992; NCTM, 2000). Thus, effective instruction should enable students to investigate the connections between various concepts and topics within mathematics. The use of concept maps can provide one avenue for a teacher to emphasize this often neglected learning objective in a way that actively engages students in constructing and communicating the depth of their knowledge visually. While concepts maps cannot be considered a comprehensive means of assessing a student's understanding of a particular body of mathematical topics, they do provide a unique view into each student's interpretation of the material. The use of concept maps offers numerous benefits to students and teachers: they can involve significant mathematics in a wide range of introductory level to upper division courses; allow for individual differences in the organization of the terms and in expressing the connections between terms; can be motivating to students; and provide the teacher with a unique view of student thinking (Bolte, 1999a, 1999b; Williams, 2002).

2 An Overview of Implementation in the Mathematics Classroom

Concept maps can be used for various purposes within a mathematics course. When used at the beginning of a term, they serve as a review of prerequisite knowledge. For example, terms related to the concept of function can be used early in a Calculus course to assist students in building a strong foundation for the remainder of the course. When used during a unit of instruction, such as investigating the properties of polygons, the construction of concept maps is a learning strategy that promotes students' understanding by identifying potential misconceptions and by focusing attention on the connections between related topics/terms. If used as a final project, for example in a proof-based geometry course as discussed below, a concept map is not only a learning tool, but also an assessment instrument that provides the teacher and student with valuable information related to specific topics and to student's ability to communicate the interrelatedness of the topics. In addition, it facilitates the evaluation of instruction with respect to designing activities which foster the development of an integrated body of mathematical knowledge.

The choice of terms to be used in a concept map may be either teacher-selected, student-generated where all terms are used by the entire class, or student-generated individually. Each of these options provides slightly different information with respect to the learner. Use of teacher-selected terms is the most directed approach and has the advantage of focusing students on the critical topics; this is beneficial when reviewing prerequisite material. Student-generated terms used by the entire class allows students to identify critical terms and ensures all students attempt to utilize the body of information. This approach is effective when students are developing an understanding of the material or as a summative project. Individual student-generated terms is the most open-ended approach. Although it provides the clearest picture of individual construction of knowledge, it is sometimes difficult for a student to generate a list that captures his/her depth of understanding. Depending on the intended purpose of the activity and the level of proficiency of students, concept maps can be completed individually, with a partner, or as a class. If completed as part of a team effort, a class discussion of the final maps which focuses on the similarities and differences between the maps can broaden students' perspectives and give students the opportunity to reflect on relationships they may not have perceived. When used in conjunction with an interpretive essay, students may extend and/or clarify the relationships illustrated in their concept maps and refine their thinking as they communicate their ideas in writing (Bolte, 2001).

3 Example of Use in a Survey of Geometries Course

An examination of the concept maps constructed as a final project in a recent geometry course provides specific examples of how students' depth of understanding and level of geometric reasoning is communicated via a concept map. The project offers insight into possible adaptations to instruction in order to address student misconceptions and omissions that became evident when analyzing the concept maps. Survey of Geometries is an upper division course that introduces students to various finite and infinite geometries, both Euclidean and non-Euclidean, through the development of axiomatic systems. With regard to the van Hiele categorization of Levels 1 through 5 of geometric concept development – Visualization, Analysis, Informal Deduction, Deduction, and Rigor (Clements, D., 2003; Fuys, Geddes, & Tischler, 1988), most students begin the course with some facility working deductively within the Euclidean system (Level 4). The course is designed to provide students with activities that enable them to progress to Level 5, Rigor, which entails comparing and contrasting different axiomatic systems.

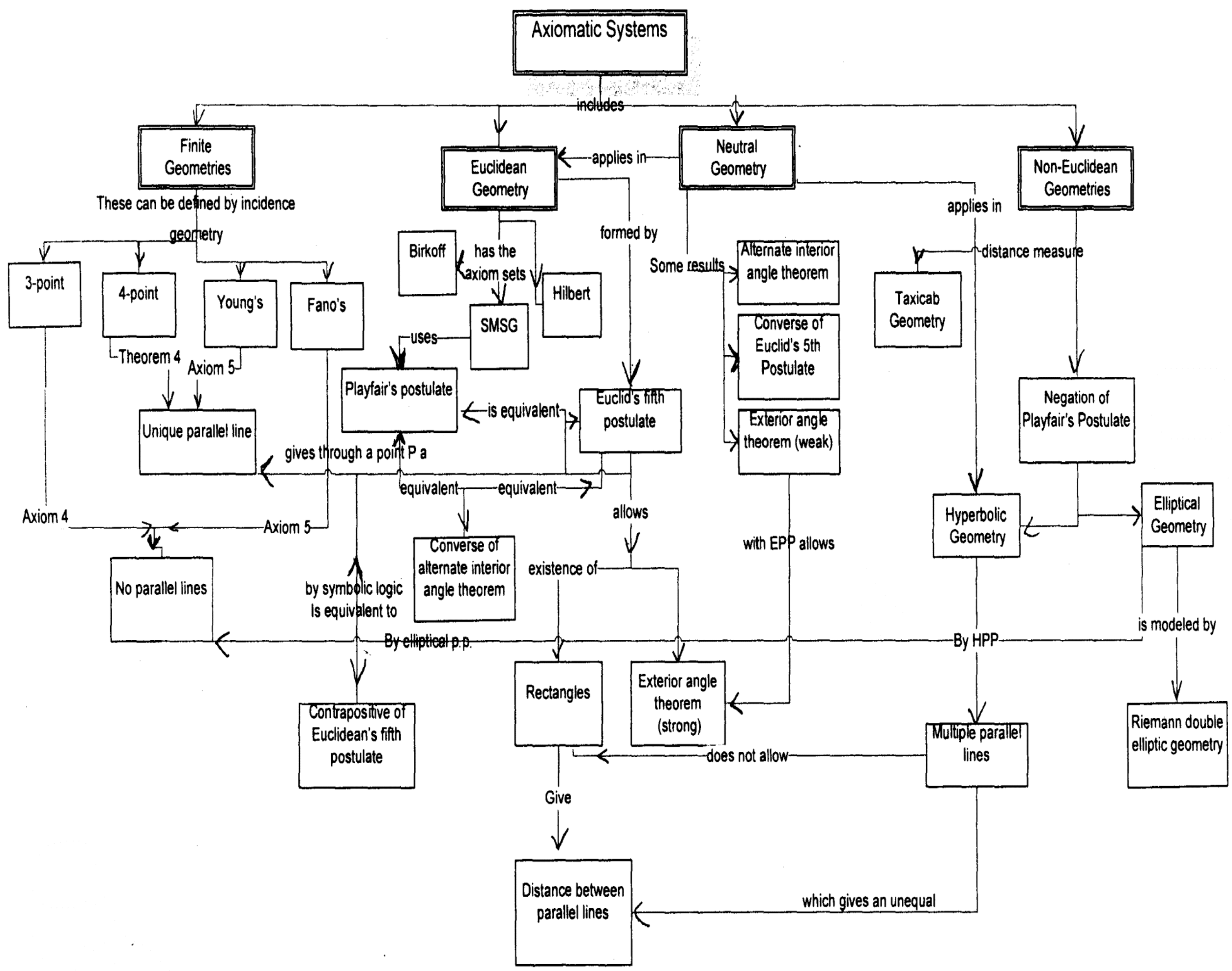
Figure 1 illustrates the development of teacher-generated terms used in this course over a period of several years. The terms used initially are listed in the first two columns. Those shown in italics were subsequently added to focus students' attention more clearly on equivalent statements and properties of each axiomatic system that were deductively proved within each axiomatic system. The bulleted terms reflect additions in response to several misconceptions and omissions that were evident in the evaluation of the most recent student work. The concept map was used in conjunction with an interpretive essay in which students expanded on the relationships indicated on the map. Consequently, the linking words used on the map did not necessarily form complete sentences or thoughts because students prefer to keep the map uncluttered and often rely on the essay for expansion and clarification.

SURVEY OF GEOMETRIES

Euclidean	incidence	*distance between parallel lines*	*Taxicab*
axiomatic system	3-point	*Riemann double elliptic geometry*	*angle sum of triangle*
neutral	Fano's	*Converse Of Alternate Interior Angle Theorem*	*rectangle*
elliptic	SMSG	*Alternate Interior Angle Theorem*	• Klein single elliptic
no parallel lines	Birkhoff	*Converse Of Euclid's 5^{th} Postulate*	• 5-point
multiple parallel lines	Hilbert	*Exterior Angle Theorem*	• Poincare half-plane
unique parallel line	Young's	*Negation Of Playfair's Postulate*	• 4-line
hyperbolic	4-point	*Contrapositive Of Euclid's 5^{th}*	• angle of parallelism
infinite	finite	*Playfair's postulate*	• Poincare disc

Figure 1. Sample of recommended terms for geometry concept map.

While Clements states "the van Hiele theory lacks detailed descriptions of students' thinking" (2003, p. 154), general descriptors are noted by Fuys, Geddes, and Tischler (1988). At the Deduction Level these include establishes interrelationships among networks of theorems, examines effects of changing an initial postulate in a logical sequence, establishes a general principle that unifies several different theorems; at the Rigor Level descriptors include explores how changes in axioms affect the resulting geometry, establishes theorems in different axiomatic systems, compares axiomatic systems. Based on indications of student thinking consistent with the relevant descriptors, analysis of student work indicates three levels of depth of understanding and integration of related topics that span two levels of geometric reasoning, Deduction and Rigor. At the most basic level, students were able to categorize different geometries according to the parallel property (i.e., no parallel lines, unique parallel lines, multiple parallel lines) that was either stated as part of the axiomatic system or deduced from the axioms. Most students correctly distinguished between finite and infinite geometries, with the exception of Young's geometry. The most common misconception noted was inferring a system contained an infinite number of points because the existence of a specific number of points was not stated in the axioms. Hence, Young's geometry was grouped with SMSG, Hilbert, and Birkhoff and linked to Euclidean by the word "are". Clusters shown in these portions of the concept maps represent the ability to reason at the Level 4, Deduction. An intermediate level of understanding is indicated in a concept map by addressing the role of statements equivalent to Euclid's 5th postulate, its negation, and its converse; and distinguishing between neutral geometry and elliptic geometry. The most common misconception at this level involved the converse of Euclid's 5th postulate, which is true in neutral geometry, and the role the Alternate Interior Angle Theorem plays in defining neutral geometry (i.e., it guarantees the existence of parallel lines). These portions of the concept maps indicate the beginning stages of reasoning at Level 5, Rigor. The most integrated level of understanding accurately conveys the relationship between the parallel postulate, the existence of rectangles, the distance between parallel lines, and the sum of the interior angles of a triangle. One example of this depth of understanding is illustrated in the map shown in Figure 2; this organization represents some proficiency at the Rigor level. Although many students correctly linked rectangles and a angle sum of 180° to Euclidean geometry, use of linking works such as "has" or "exist in" rather than explicitly indicating the deduction of these properties from a parallel postulate indicated a more superficial understanding of the relationships. In general, it appears the use of descriptive, mathematically meaningful linking words is one factor that indicates the depth of student thinking.



3-point Geometry, axiom 4 - Two distinct lines are on at least one common point.
4-point Geometry theorem 4 - Each distinct line has exactly one line parallel to it.
Fano's Geometry, axiom 5 - There is at least one point on any two distinct lines.
Young's Geometry, axiom 5 - For each line l and each point P not on l, there exists exactly one line on P that does not contain any points on l.

Figure 2. Student constructed Survey of Geometries map.

Analysis of the concept maps with respect to the level of geometric reasoning represented, student misconceptions and omissions, and the level of integration illustrated in the concept maps underscores the need for instruction to focus on the role of deductive reasoning in determining properties of given axiomatic systems and the effect of changing given axioms. For example, altering the Euclidean Parallel Postulate results in hyperbolic and elliptic geometries, altering the definition of the distance between two points results in Taxicab geometry, altering the notion of a line being infinite and unbounded to infinite and bounded yields elliptic geometries, and defining polar points (diametrically opposite points on a Euclidean sphere) as a single point creates an incidence geometry with the elliptic parallel property. The inclusion of the additional terms noted earlier provides examples and non-examples of finite geometries with certain properties (e.g., incidence geometry and hyperbolic parallel property), two models of hyperbolic geometry that are embedded in Euclidean geometry, and a concept that provides an equivalent statement of the hyperbolic parallel property. Facility with these terms would further indicate an ability to reason at the Rigor level.

4 Summary

Most students consider the project worthwhile, indicating it deepened their understanding of mathematics; however, some students continue to indicate they prefer more computational, less open-ended assignments that require less commitment. The use of an accompanying interpretive essay helps some students refine the ideas expressed on their map, while for others it appears to hinder the complete development of their concept map. From the teacher's perspective, concept maps provide an explicit means of discerning the depth of student thinking. Specifically with respect to a proof-based geometry course, this project models the last phase of instruction, Integration, required for students to progress from one level to the next in the van Hiele model; during this phase the "student summarizes all that he/she has learned about the subject, then reflects on his/her actions and obtains an overview of the newly formed network of relations now available" (Fuys, Geddes, & Tischler, 1988, p. 7).

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