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USING CONCEPT MAPPING IN THE DEVELOPMENT OF THE CONCEPT OF POSITIONAL SYSTEM

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Abstract. Psychologists such as Vygotsky and Skemp indicate that as a superordinate concept the understanding of positional system requires knowledge of several bases for its adequate development. However, current elementary mathematics curricula fail to adequately develop the concept of positional system, attempting instead to teach operations in base ten in isolation. This paper exhibits the power of concept mapping to reveal to teachers the centrality of this concept in elementary mathematics. The map presented here, constructed by Meridith, a pre-service teacher, also features the pedagogical content knowledge required to successfully teach the concept of positional system and the other mathematics concepts to which it is related. Meridith's pedagogical treatment is neither simplistic nor reductionist, but reveals the conceptual essence of the concepts in question and the complexity of their relationships within elementary mathematics when taught as a conceptual system.

1 Introduction

The teaching of multiple bases to develop the concept of positional system that figured so prominently in the US mathematics education reform during the 1960s and 1970s was swept out of favor with the advent of the back to basics movement that succeeded the "new math" era. Unfortunately, its former prominence in the elementary mathematics curriculum has yet to be restored. It is not included in the current reform effort of the National Council of Teachers of Mathematics (NCTM, 1989, 2000), and is absent from most elementary school mathematics and elementary mathematics methods textbooks as well (cf. Burris, 2005, for a notable exception). The multi-base blocks invented by Zoltan Dienes some fifty years ago have virtually disappeared not only from US classrooms but from the catalogs of suppliers of mathematics manipulatives as well (the Prairie Rainbow Blocks developed by George Gagnon constitute a rare exception). Only base ten blocks are in common use in US classrooms. And yet not only Dienes, but Skemp (1987) and Vygotsky (1962) stressed the importance of teaching multiple basic level concepts for the formation of a superordinate concept such as positional system.

> As long as the child operates with the decimal system without having become conscious of it as such, he has not mastered the system but is, on the contrary, bound by it. When he becomes able to view it as a particular instance of the wider concept of a scale of notation, he can operate deliberately with this or any other numerical system. The ability to shift at will from one system to another…is the criterion of this new level of consciousness, because it indicates the existence of a general concept of a system of numeration (Vygotsky, 1962, p. 115).

Skemp (1987) stresses that a superordinate concept can not be developed in the mind on the basis of a single basic level concept. At least several basic level concepts are required. A concept of positional system can not be developed through the teaching of base ten alone. And Vygotsky (1962) asserts that a child can not master the decimal system without attaining a mastery of the more general concept of positional system. Thus, as these leading psychologists in the theory of learning attest, one can not understand base ten without learning other bases as well.

2 Developing the Concept of a Positional System in Teacher Education

Not only is the knowledge of multiple bases essential for understanding the concept of a positional system, but the concept itself provides a conceptual foundation for the four fundamental operations on whole numbers and for the development of the concept of variable, exponent, polynomial and polynomial operations, decimals, fractions, and area and volume experienced through the geometric modeling of positional systems. Consequently, the omission of this important means to concept development has multiple consequences for the learning of mathematics beyond the elementary level.

While the case for teaching multiple bases has been well documented in the psychological literature (Vygotsky, 1962; Skemp, 1987), its omission from elementary school mathematics textbooks presents difficulties for prospective and practicing teachers who may acknowledge its importance but be inclined to protest that "it's not in

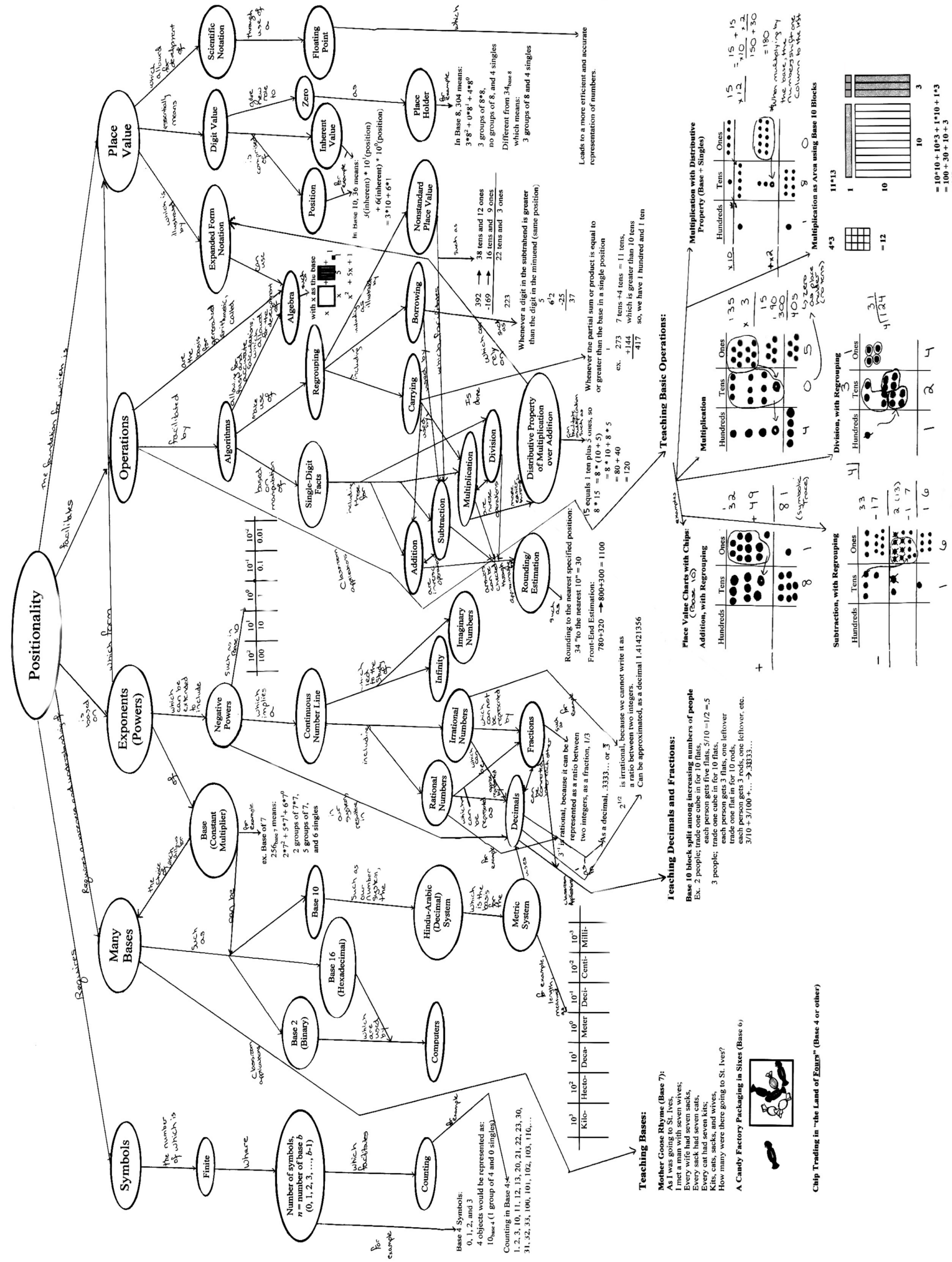

Figure 1. Meridith's concept map.

the curriculum”. Assigning pre-service and in-service teachers the task of constructing a concept map exploring the connections of the concept of positional system to other important mathematical concepts, reveals the centrality of this concept both in elementary mathematics and as a foundation for concepts encountered at the middle school level and beyond. The concept map shown in Figure 1 was made by a pre-service teacher in response to such an assignment. When the concept of positionality was addressed in her elementary mathematics methods course, concept mapping was also introduced, and students were asked to begin to draw a map centered on the role of positionality in school mathematics. As additional mathematics concepts were explored, students were encouraged to continue adding to their maps, so that the construction of the concept map became a project that continued to develop over the semester. The assignment revealed to students the multiple connections the concept of positional system has with mathematics concepts that children will study in the later elementary and middle school years and even beyond. These include such concepts as decimals, exponentials, area, volume, and polynomials and their operations.

3 Meridith’s Map

Figure 1 presents a view of Meridith’s concept map in its entirety. Figure 2 focuses on the extreme left section of the map, which reveals her understanding of the need to establish the concept of positional system on the foundation of multiple bases. Under the designation “Symbols” we see that she understands that for any base b, numerals from 0 through 1, 2, 3, …, b-1 are necessary to designate the numbers in the system. Base 4 is provided as an example, with 0, 1, 2, and 3 as numerical symbols, and counting in base 4 is illustrated.

Moving to the right across the map (Figure 2), Meridith further develops these ideas, beginning with the initial proposition that the concept of “positionality requires awareness and understanding of many bases”, including base 2 and 16 used in computers, and base ten employed in both the metric and Hindu Arabic numeration systems. Her illustrations of the powers of ten designated in the metric system and the powers of 7 in the expanded form of the number 256 $_{base\ 7}$ show the link to exponentiation that derives from a consideration of meaning of numbers across systems employing diverse bases.

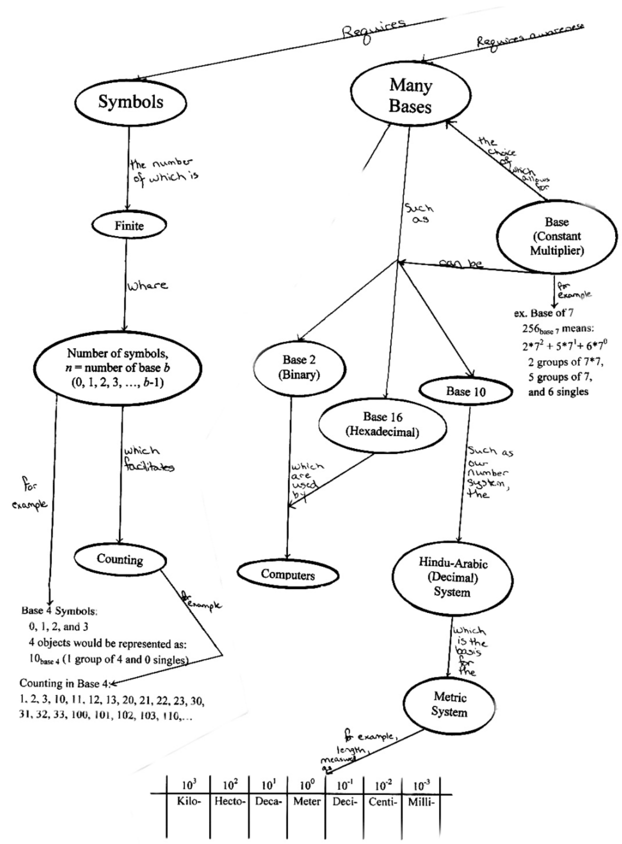

Figure 2. Left side of the Meridith’s concept map showing “Symbols” and “Many Bases” as subsumed concepts.

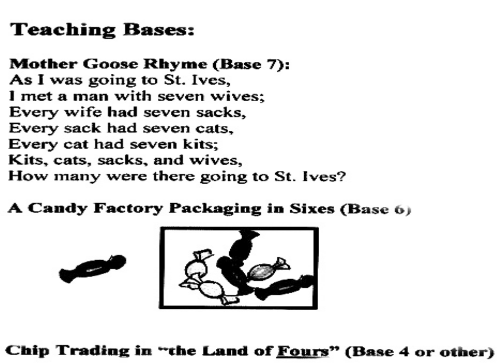

Figure 3. Classroom applications involving bases.

Figure 3 identifies the section of Meridith’s concept map that includes three classroom applications involving different bases. A Mother Goose rhyme in base 7, a packaging example in base 6, and chip trading in base 4, are examples of methods that can be used with elementary students to develop the concept of positional system from the study of multiple bases. Pre-service teachers were encouraged to include pedagogical methods (and these are found throughout Meridith’s map), as exemplary pedagogical practices that embody important conceptual content.

Chip trading in base 4 is treated in Burris' (2005) text for elementary teachers *Understanding the Math You Teach*, a feature that renders this text unique. The use of chip trading to teach the fundamental operations of addition, subtraction, multiplication, and division appears in the bottom right section of Meridith's concept map, denoted as "Teaching Basic Operations" (see Figure 1). This section is specifically highlighted in Figure 6 and represents a pedagogical approach that connects the algorithmic operations with their conceptual genesis.

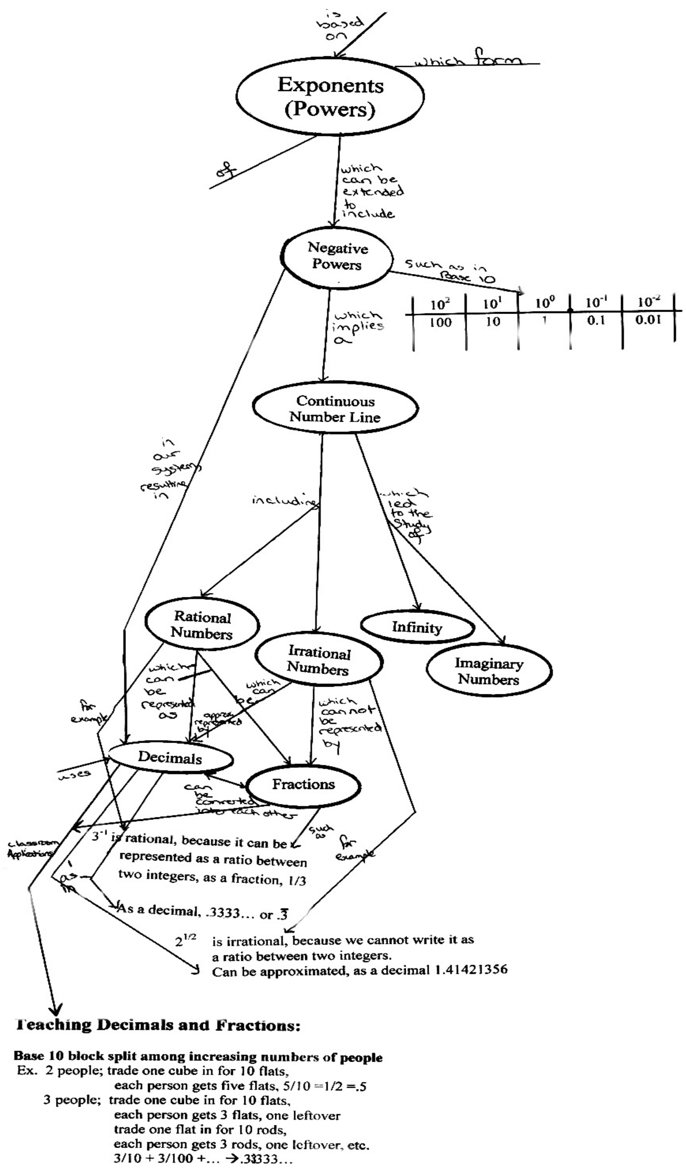

Figure 4. Center of Meridith's concept map showing "Exponents" as a related concept.

Considerations of exponentiation are shown in Figure 4, including the extension to negative integer and fractional powers, which designate fractional and irrational numbers, respectively, and to their repeating (rational) and infinite non-repeating (irrational) decimal representations. Meridith goes further to relate to the real number line the combined infinite set of all numbers capable of being expressed as a base raised to an integer or fractional power, and illustrates the generating of terminating and repeating decimals from partitive division using base ten blocks, an effective teaching strategy.

Figure 5 shows the interrelationships among the four fundamental operations of addition, subtraction, multiplication, and division of whole numbers, together with attendant processes of regrouping as required. Embedded within Figure 5 is an important link between the standard algorithms for these processes and modeling of similar operations with polynomials. The standard algorithms are unfortunately downplayed in the current reform movement (NCTM, 1998) but are, as Meridith's map reveals, fully conceptual. She illustrates the ease of transition to algebra from a consideration of multiple bases, as it is an easy step from the notion of a variable base b, where b can be any integer greater than 1, to the variable x which can be any real number. One simply removes the positive integer restriction to produce a real valued variable. Further, the familiar expanded form for numbers in various bases (such as 304 $_{\text{base } 8}$ shown here and 256 $_{\text{base } 7}$ in Figure 3) is isomorphic to the form in which a polynomial (such as $x^2 + 5x + 1$) must of necessity be written, since the value of x is unknown and hence, its terms cannot be added together. Meridith uses an area model for the polynomial. In addition, she integrates both estimation by rounding and scientific notation into this section of the map.

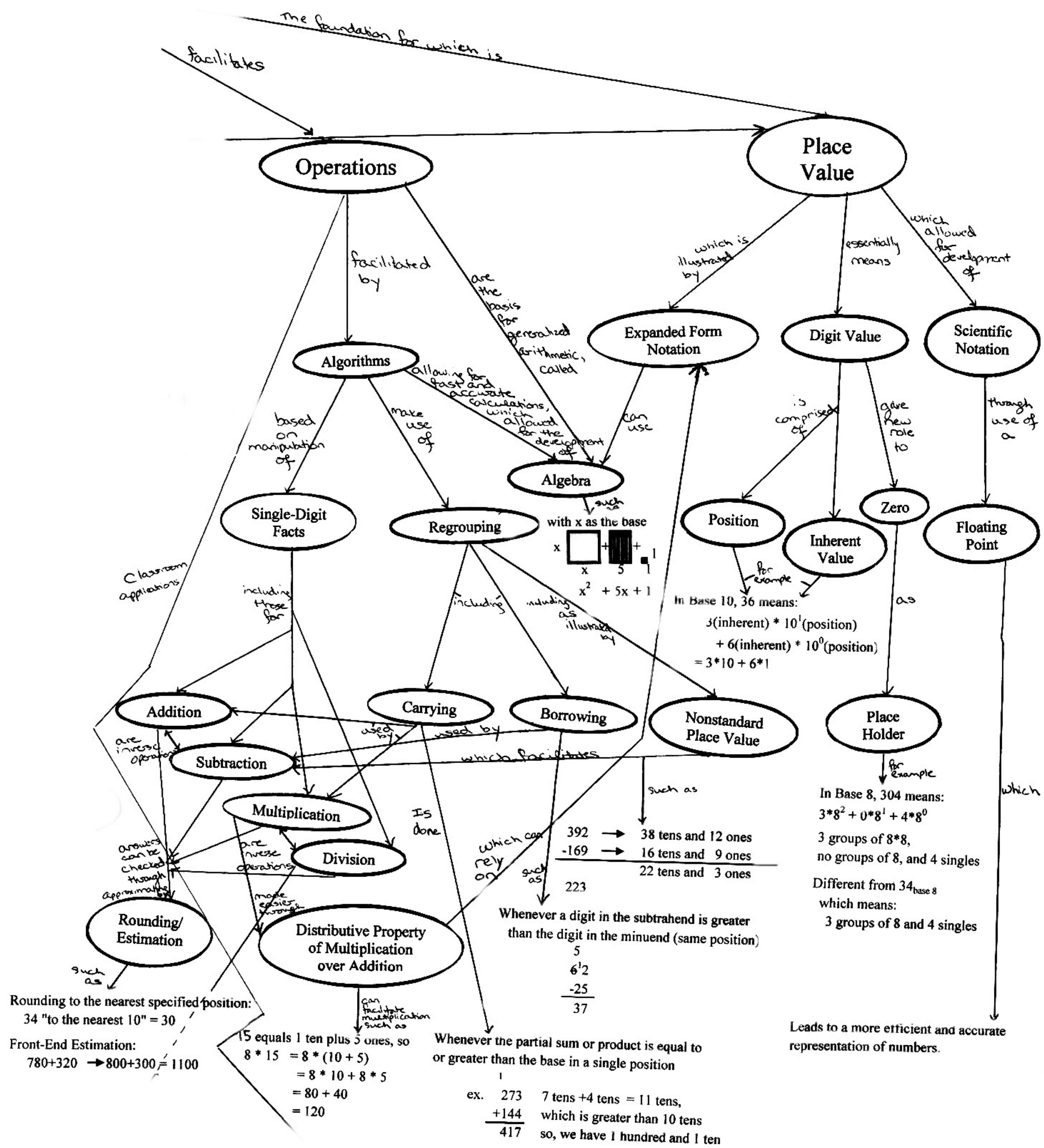

Figure 5. Right side of Meridith's concept map showing "Operations" and "Place Value" as subsumed concepts.

In Figure 6, Meridith reveals the pedagogical content knowledge necessary for proper instruction in positionality employing chip trading (Davidson, Galton, & Fair, 1975) with the requisite trades for regrouping as required in performing the four fundamental multi-digit operations on integers. She also employs an area model for multiplication. It is noteworthy that the conceptual and the algorithmic are connected without a separation of "concepts" and "procedures".

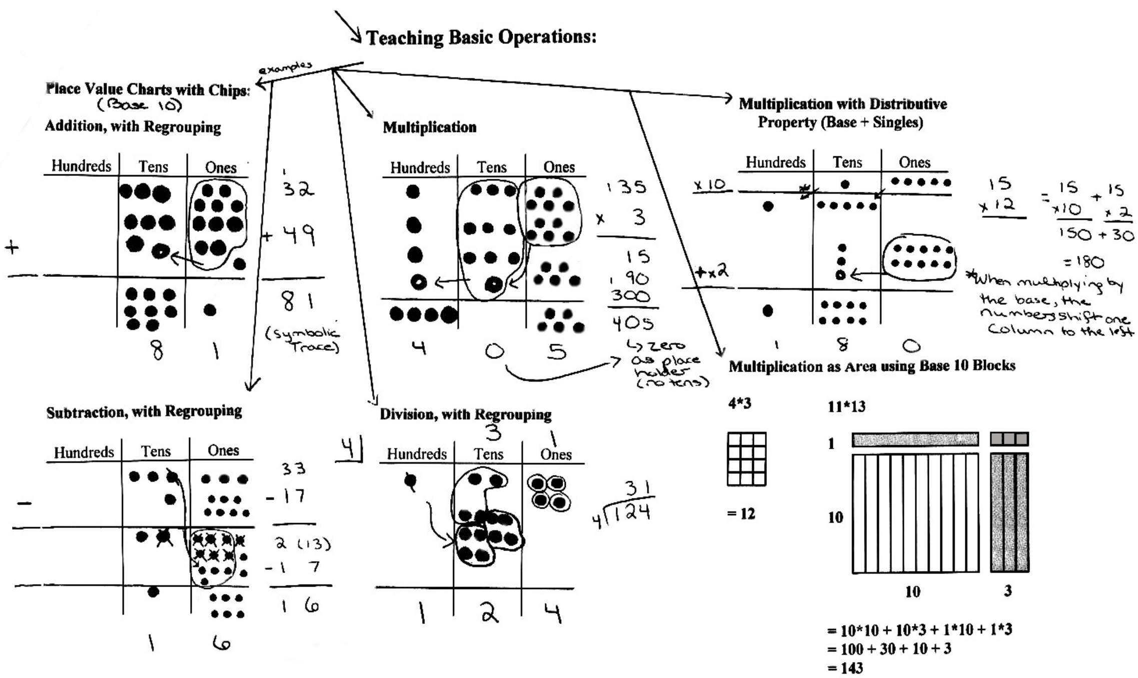

Figure 6. Meridith's pedagogical approach connecting the algorithmic with the conceptual.

Under "Teaching Basic Operations" the use of chip trading as a pedagogical tool is displayed. In the "Addition with Regrouping" model, Meridith illustrates the removal of ten "ones" chips and their trade for a "ten" chip in adding 32 + 49 = 81. In the "Subtraction with Regrouping" model, she shows the manner in which a "ten" chip is traded for ten "ones" chips, and then 7 "ones" chips are removed from the resulting 13 "ones" chips. Then one "ten" chip is removed from the remaining two "tens" chips in the subtraction problem 33 -17 = 16.

In the depiction of multiplication using chip trading, Meridith solves the problem 135 x 3 = 405 by tripling the 5 "ones" chips and trading ten of the resulting 15 chips for a "ten" chip, then tripling the 3 "tens" chips and trading the resulting 9 +1 "tens" chips for a single "hundred" chip. Tripling the original "hundred" chip and adding this additional "hundred" chip results in the solution of 4 "hundreds", no "tens" and 5 "ones" chips. She then shows how in multiplying 15 x 12, the 12 is first split into 10 + 2, then each addend separately multiplies 15, and finally the resulting partial products are added to obtain 180. The distributive property is the underlying conceptual mechanism here, and this is emphasized in the area model for multiplication of 13 x 11.

Finally, Meridith illustrates the use of chip trading in the division of 124 by 4. First the "hundred" chip is traded for 10 "tens" chips to provide 12 "tens" chips. Twelve tens divided by 4 is 3 tens, and finally four "ones" divided by four is 1 "one". Therefore the quotient is 31 which is 3 tens and 1 unit.

The power of chip trading is that it requires the child to employ the same cognitive processes that are required for computing efficiently with multi-digit numbers in our base ten positional system. As these processes are employed, the conceptual content of the algorithms is continually reinforced; they are not arbitrary “rules” to be memorized and executed by rote, but meaningful processes that rely for their power on the concept of positional system and the properties of actions such as the distributive property of multiplication over addition. Eventually, children have no need for the chips and simply invoke the representation of the trades. Finally, they are able to perform the requisite mathematical actions on the numbers alone, and to see that the algorithms are merely the symbolic trace of the meaningful mathematical actions they formerly performed on objects. When they attain this level of competence and understanding they will not find actions on large numbers (which would be too cumbersome to be easily performed with objects) to be daunting, nor will they need a calculator to perform them.

Meridith’s map thus shows connections of the concept of positional system with its genesis from a consideration of multiple bases (perhaps begun with simple nursery rhymes, and pictures for young children as shown in Figure 3), through connections with exponentiation, decimals, fractions, the real number line, estimation and rounding, the metric system and scientific notation so important for measurement, area models, the concept of variable, and the operations of addition, subtraction, multiplication, and division of both integers and polynomials.

4 “Death by Decimal”

Concept mapping reveals the centrality of the concept of positional system in the conceptually dense system of concepts that comprise elementary school mathematics. Not only does it connect to many important concepts that students will study concurrently or for which it will prepare them for study in the future, it is also a prerequisite for any real understanding of the base ten system. And the consequences of failure to adequately grasp this concept in real world applications range from measurement inaccuracies in trades such as construction to those in professions such as medicine. The first can be costly; the second deadly.

In her study of the mathematical errors of student nurses, Pirie (1987) documents the extent to which student nurses fail to correctly use mathematics to make mindful decisions in such tasks as unit conversions, dosage calculations, and fluid monitoring. Fragile and/or incorrect conceptual development in mathematics often invites the use of procedural shortcuts that increase the potential for error and the possibility of disastrous results for a patient whose life may depend on the correctness of the calculation. In the absence of fundamental conceptual grounding, the same mathematical procedures that could be used to promote the health and well-being of a patient become unreliable.

The severity of miscalculation is evident in the simulation study published in the *American Journal of the Diseases of Children*. Perlstein et al., whose study involved the staff of a neonatal intensive care unit working with simulated physicians’ orders, reported that "56% of the errors tabulated would have resulted in administered doses ten times greater or less than the ordered dose" (sited in Pirie, 1987, p.145.) Lesar (2003) studied and classified the 200 tenfold errors in medication dosing that occurred in an 18 month period at a 631-bed teaching hospital citing such errors as a misplaced decimal point, adding an extra zero, or omitting a necessary zero. Przybycien (2005, p. 32) reports that a physician ordered morphine .5 mg IV for a 9-month-old baby but because of a missed decimal an inexperienced nurse gave the baby 5 mg of morphine IV, and the baby died. It is an example of “death by decimal” and the lack of meaningful understanding of positionality continues to lead to such tragedy.

5 Summary

Using concept mapping in the development of the concept of positional system substantively reveals the need to teach multiple basic level concepts for the formation of a superordinate concept. Meridith’s carefully considered concept map of positionality is in sharp contrast to the superficial treatment often found in popular mathematics texts which display a decimal number with digits to the right and left of the decimal point labeled to indicate the name and relative value of each position. Meridith’s concept map clearly indicates what must be taught to students for a meaningful understanding of positionality to develop. It reflects her in-depth exploration of the meanings associated with and underlying the concept of positional system, its antecedent concepts, and the complexity of their interrelationships within a conceptual system. The map addresses a serious deficiency in current elementary mathematics programs and provides a reliable direction for future mathematics curriculum development. The inadequate development of positional system inhibits future learning in mathematics and has important consequences for societal applications that require knowledge of the decimal system. Research in the field of medicine corroborates the number of deadly errors attributable to misplaced decimals in fluid monitoring and drug dosage calculations. It would be difficult to imagine a nursing student who understands the concepts and relationships depicted in Meridith's map ever making the devastating errors chronicled in Pirie's (1987) and Przybycien's (2005) research.

6 Acknowledgement

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References

Burris, A. (2005). *Understanding the math you teach.* Prentice Hall: Upper Saddle River, NJ.

Davidson, P.S., Galton, G.K., & Fair, A.W. (1975). *Chip trading activities.* Fort Collins, CO: Scott Resources.

Lesar, T.S. (2003). Tenfold medication dose prescribing errors. *The American Journal for Nurse Practictioners, 7*(2), 31-32, 34-38, 43.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (1998). *The teaching and learning of algorithms in school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Pirie, S. (1987). *Nurses and mathematics: Deficiences in basic mathematical skills among nurses.* London: Royal College of Nursing.

Prairie Rainbow Blocks, Prairie Rainbow Company. Oakland: CA.

Przybycien, P. (2005). *Safe meds: An interactive guide to safe medication practice.* St. Louis: Elsevier Mosby.

Skemp, R. (1987). *The psychology of learning mathematics.* Mahwah, NJ: Lawrence Erlbaum.

Vygotsky, L.S. (1962). *Thought and language.* Cambridge, MA: MIT Press.