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CONCEPT MAPS AS RESEARCH TOOL IN MATHEMATICS EDUCATION

Barbro Grevholm
University of Agder, Norway

Abstract. Using concept maps as research tools in different ways has been found productive in a longitudinal study on student teachers' development of mathematical concepts. Maps were used both in a priori analyses of groups of concepts, that are part of content in courses, and as a tool for students to express their conceptions of functions, equations and so on. Examples of one student's maps drawn at three different occasions over a time period of 15 months reveal that the concept image is developing over time although the student is not studying mathematics or working with mathematics. The slow development of concepts and need for maturation and cognitive processes to take place is illustrated by the examples. Some outcomes of the study are mentioned and indicate further use of concept maps. Mathematics teachers need for development of a professional language is one important result and a model for mathematics teacher education seen as the development of the professional identity another result.

1 Introduction

Longstanding work with mathematics teacher education has stimulated my curiosity in how student teachers develop concepts in mathematics and mathematics education. In 1996 a new teacher education programme started in Kristianstad University for prospective teachers in mathematics and science for school years 4-9. I led the work with creating the mathematics courses for this programme (Grevholm, 1998), building on all my earlier knowledge and experience from teacher education. Consequently, I was eager to follow the development of the education and its outcomes. I decided to carry out a longitudinal research study and in this try to focus on student teachers' conceptual development, as I (as many other researchers) had noticed how important this is for the in depth learning. In Sweden no such study was conducted earlier.

It is complicated to do research on students' conceptual development as the concepts an individual holds are not open to direct observation. They can be studied only indirectly through actions, statements or answers given by the individual student. Thus I started out to collect all data that could possibly help me to observe students' conceptual development. Soon after I had started the data collection, Joseph Novak came to visit our department. When he heard about my study he tried to convince me that it should be productive to use concept maps. At the beginning I hesitated. From the examples I saw of biological concepts it was obvious that objects and events could be studied and observations formulated in knowledge propositions and represented in concept maps. You could for example study a plant and describe the development. But I did not find it possible to look at 'objects and events', when it came to mathematics. All mathematical concepts are abstract. It took me some time to reflect more closely on the differences between concepts in science and in mathematics. After a while I came to the conclusion that objects in mathematics can mean the mathematical objects like numbers, shapes, equations, functions, and expressions and so on. And events can be seen as the operations, processes, constructions or actions we make with these objects. After this interpretation it became clear to me how useful the concept maps could be also in my study. Based on experiences from the study, the use of concept maps as a tool for research on development of mathematical concepts is explored and discussed in this paper.

In this paper I want to explore the use of concept maps in mathematics education research and try to answer the questions:

- Can concept maps as a tool in research contribute to our understanding of students' conceptual development in mathematics?
- In what ways can concept maps be useful?

2 Theoretical background

Theories in mathematics education deal with phenomena such as meaningful learning versus rote learning, conceptual knowledge or procedural knowledge, mathematical phenomena seen as procedures or objects, and conceptual change and development as an important part of learning (Ausubel, 1963, 2001; Ausubel, Novak & Hanesian, 1978; Hiebert & Lefevre, 1986; Sfard, 1991; Tall & Vinner, 1981). In several of these theories mathematical concepts and the development of concepts are crucial.

Knowledge construction is a complex product of the human capacity to build meaning, cultural context, and evolutionary changes in relevant knowledge structures and tools for acquiring new knowledge, according to Novak (1993). Novak claims that concepts play a central role in both psychology of learning and in epistemology. In his Human constructivism Novak (1993) builds on Ausubel's assimilation theory of learning to describe the process by which humans engage in meaningful learning. Two key ideas in assimilation theory are *progressive differentiation* and *integrative reconciliation*. Novak explains that as new concepts are linked nonarbitrarily to an individual's cognitive structure progressive differentiation occurs. The integrative reconciliation occurs when groups of concepts are seen in new relationships.

Hiebert and Lefevre (1986) devoted much interest to the discussion of conceptual and procedural knowledge. Conceptual knowledge is equated with connected networks. Conceptual knowledge is knowledge that is rich in relationships. Procedural knowledge is a sequence of actions. Sfard’s reification theory (1991) concerns mathematical phenomena seen as processes or as objects. In an often quoted paper Tall and Vinner (1981) discussed *concept image* and *concept definition*. They note that many concepts are not formally defined at all but we learn to recognise them by experience and usage in appropriate contexts. After some time the concept may be refined in its meaning and interpreted with increasing subtlety. They continue:

> *Usually in this process the concept is given a symbol or name which enables it to be communicated and aids in its mental manipulation. But the total cognitive structure which colours the meaning of the concept is far greater than the evocation of a single symbol. It is more than any mental picture, be it pictorial, symbolic or otherwise. During the mental processes of recalling and manipulating a concept, many associated processes are brought into play consciously and unconsciously affecting usage and meaning (p. 152).*

At this stage they introduce the term concept image (held by the individual) to describe the total cognitive structure that is associated with the concept. The concept image is personal and changes when the person meets new stimuli and matures. Tall and Vinner (1981) also make a difference between the formal concept definition (accepted by the mathematical community) and the personal concept definition (the words the student uses for his own explanation).

Other researchers, such as Vollrath (1994), have discussed the meaning and development of concepts. He claimed that the student reaches stages of understanding and that there is no final understanding. Ausubel (1963) contrasts meaningful learning to rote learning, where meaningful learning results in the creation and assimilation of new knowledge structures. In many of the theories there seems to be a continuum from lower quality learning to higher quality learning, where higher quality often includes concept development. The theories can be seen as different ways to model the quality of learning and how it evolves.

Joseph Novak (Novak & Gowin, 1984) has introduced *concept maps* as a cognitive tool and as a research tool. In his case the researcher drew the maps in order to give a concentrated representation of what an interviewee answered (Novak, 1998). The map is used for data reduction and concentration of content. From his writings (Novak, 1998) it is well known how he defines a concept and what he means by a concept map. In Novak’s maps it is important that the map is built of knowledge propositions. The nodes that contain concepts should be connected by linking words to form propositions, which represent knowledge sentences. Normally the map is also hierarchical. The map can be seen as a picture or image that the learner chooses to draw from what he experiences as the mental representation of his knowledge. A map drawn by a student is time-dependent, individual and dynamic.

In research literature many different sorts of concept maps have been introduced (see for example Williams, 1998). A concept map differs from for example a mind map, which is a looser construction and does not necessarily show how the learner wants to draw his knowledge representation. A spider web map has no hierarchical character. In this paper I use Novak’s definition of concept and concept map.

Research using concept maps as a tool

Williams (1998) used concept maps to assess the conceptual knowledge of function. She studied concept maps drawn by students and professors of mathematics and compared them. Her dissertation was based on that work and she claims that "Concept maps are a direct method of looking at the organization and structure of an individual's knowledge..." (p. 414). This strong claim can be questioned and she modifies herself in the conclusions. There she states (p. 420): "The degree to which concept maps describe a person's mental representations is, of course, impossible to know." But her final conclusions are important:

The analysis also provided information about students' understanding that is not readily gained from traditional paper-and-pen tests. Concept maps therefore, provide important information about conceptual understanding and can play a useful role in the mathematics researcher's repertoire of tools. (p. 420)

Novak and his colleagues used concept maps in many studies and argue strongly for their potential in research and in learning (Novak, 1985, 1993, 1998). In her master's research, conducted at Samoa University, Afamasaga-Fuatai (1998) used concept maps. Her research shows that students found concept maps useful in their learning and understanding of mathematics. It helped in systematic analysis of a topic for the interconnections between relevant concepts and procedures, and facilitated problem solving. There are other studies available using concept maps in research (for references see Williams, 1998 and McGowen & Tall, 1999), studies mainly in science didactics but also in mathematics didactics. One of my doctoral students have replicated parts of my study and used concept maps starting from mathematical expressions instead of concepts (Hansson, 2006).

3 The use of concept maps in the research study of student teachers' mathematics

There are different ways to use concept maps and as a tool in research I first used it in an a priori analysis (Artigue, 2002) of the expected learning of students in their course. For example I drew a map of the concept *fractions* (with 25 nodes and 30 links), where I tried to include all the important features about fractions that I consider crucial in the course the student teachers were going to study. Later, after the students had answered a questionnaire about the course in number theory, I used a concept map in the analysis for data reduction on the answers about fractions, by drawing all alternative answers given by the students and the links they proposed. In comparing my a priori map with the map constructed by the students' answers I could see what parts of the expected exposed learning that had taken place and not.

Another way I used the maps, repeatedly over time to follow the conceptual development, was to let students express their view of a concept. Below I will show some examples of the data it produced and discuss what results one can get from it. First I need to say something short about the study in which the maps were used.

4 Methods in the study of student teachers' concept development in mathematics

The method used in the study drawn upon here is mainly qualitative investigation of data from different types of documentation of students' cognitive development during a teacher preparation program. Concept maps are used as a tool both for analysing the content of the teacher education to find the fundamental concepts, to investigate students' answers in questionnaires and interviews, and for the students to express a picture of their current concept structure.

The overarching questions posed are phrased: How are the studies of mathematics and mathematics education influencing student teachers' development of concepts in these areas? How do student teachers' perceptions of and attitudes to mathematics change during the education? What impact does the development of concepts have for the learning outcome and for the students' perception of their own learning? The studied group consisted of 48 student teachers studying to become compulsory school teachers in mathematics and science for school years 4-9. I have reported on this study elsewhere (Grevholm, 1999, 2000, 2002, 2003ab, 2004, 2005, 2006) and here I am only going to discuss the use of concept maps as a tool in research.

Students' concepts are not open to direct study by the researcher. They have to be observed in an indirect way and often only in fragments. Some researchers argue that concepts should be studied through their appearance in students' actions. This is however time-consuming and a difficult process. Here questionnaires were the first attempt to get an image of students' conceptions, followed by interviews based on the answers given. The impression was that far too little of what students carry in their heads about the concept was revealed in this way. I was convinced that students could expose more to me about their concept image. At this stage concept maps were introduced as the answer form for students. As will be shown below a much richer material was retrieved in this way and substantial knowledge about how students express their mental structures through maps became available. By having students draw maps at several times with long intervals the development over time of the structures could be studied. With the examples below I want to illustrate that if concept maps are used as a tool for research, the findings differ in a positive way from results from questionnaires and interviews.

5 Examples of collected data

In the investigation an example of the outcome of the questionnaires could look like this. To the question 'What do you mean by a function?' Lina, one of the students, answered before and after the course in function theory (calculus):

1) – *y* is depending on how big *x* is. There is an infinite number of answers as you can vary *x* (January 1999).
2) – for example *y*=k*x*+m. *y* is here a function of *x*. So y is depending on the *x*-value. You can illustrate a function graphically (March 1999).

At both occasions Lina holds on to the idea that y is depending on *x*. In the first answer she talks about answers to the function, which may indicate that she perceives each calculation of the *y*-value as an answer to a problem. She cannot see a function as an object (Sfard, 1991). In the second answer she gives an example, the simplest possible function she has worked with, the linear function, although in a general form. She also adds the information that one can illustrate a function graphically. In the second answer she actually reveals more than in the first answer.

Still both these answers give very little information about the mental representation or concept image (Tall & Vinner, 1981) she has of the concept function. I was convinced that the student could show me more of her knowledge structures than these short sentences. The drawing of concept maps was already familiar to the students from other subjects in their education. Still I was aware of the fact that it is very demanding to try to draw a concept map of your own knowledge.

At the end of the course in calculus (five weeks in the sixth term of the 4.5 years long education programme) Lina together with one fellow student drew a map of the concept function. The task given was to individually draw one map of function and one of equation. The students did not follow the instructions. The map is not an individual map and it is a map of both equation and function in the same picture. From function there are seven links to equality, inequality, variables, straight line, proportionality, coordinate system and rule or instruction for calculation. Coordinate system is linked to coordinates and to x-axis and y-axis, which is an example of rather trivial facts often present in novice's maps (Williams, 1998). The map is consistent with Lina's answers in the questionnaires but contains more.

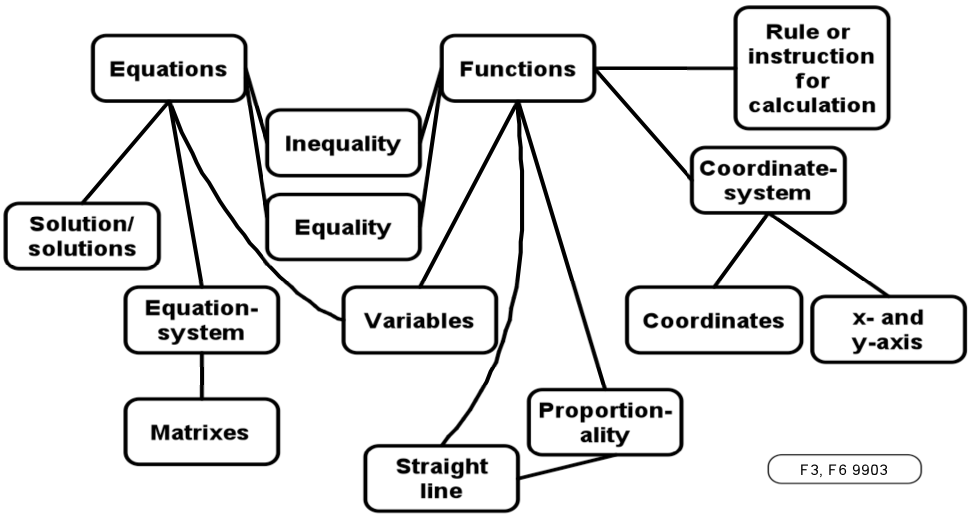

Figure 1. The first concept map of function (and equation) drawn by Lina and a fellow student in March 1999.

Nine months after the first map was drawn I met Lina again for an interview and she had been asked to draw a second map without looking at the first one (which resided with me). In the meantime Lina had studied other subjects than mathematics and she had not worked with her mathematics in organised studies at all. In spite of this it is obvious that her second map is richer than the first one. It contains more concepts and more propositions. She has removed the concepts straight line and proportionality and has added on domain, range, the properties even or odd, graph, primitive function and integral. In adding the properties she shows that progressive differentiation in her concept picture has taken place (Novak, 1993). She removes variables and co-ordinates and writes *x* and *y* instead. In the first map she talks about a rule or instruction for calculation. In the second map she gives a definition instead. She also explains that the same *y*-value can be related to different *x*-values. Still there are several unclear links in her map. Why does she connect domain and range in different ways? Why are *x* and *y* not connected to the box ‘a coordinate system’?

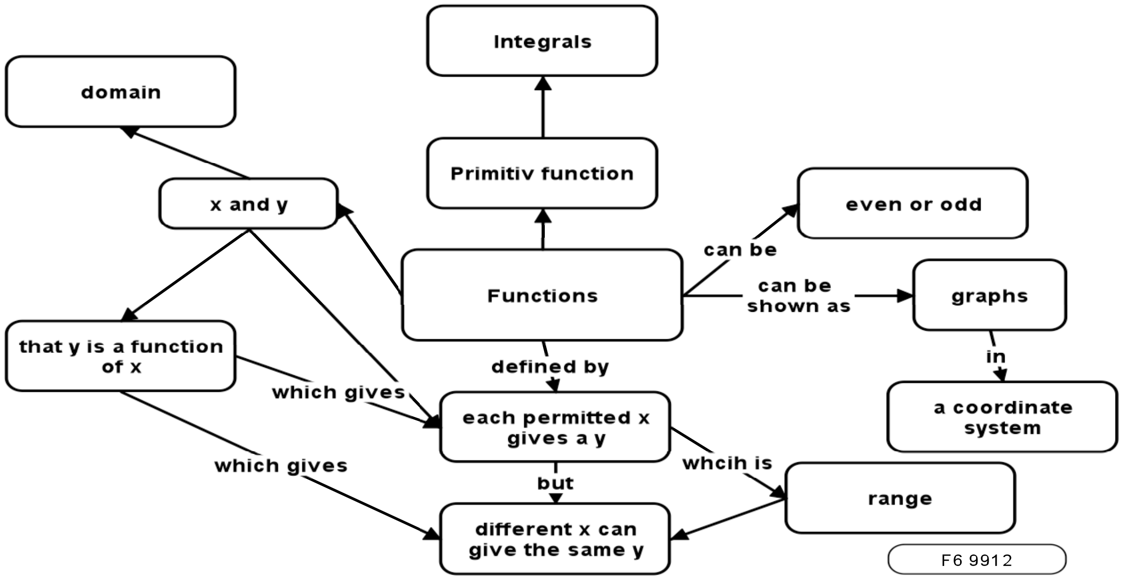

Figure 2. The second concept map of function drawn by Lina in December 1999.

Although she did not study mathematics from March 99 to December 99, changes in her concept map have taken place during that time. What the map shows is probably knowledge that has been learnt in a meaningful way (Ausubel, 1963). Otherwise it would have been forgotten and not retrievable after such a long time. Above is the second and below the third map drawn by Lina.

The third map was drawn six months after the second one, again without access to the first and second ones and without Lina having had any mathematics studies in the meantime. Again the third map is still richer than the two earlier ones. In the third map she has linked range and domain to the definition of function in a better structured way than before. This is an example of integrative reconciliation in the concept structure (Novak, 1993). Instead of talking about graphs she now mentions curves and gives a number of possible properties for them.

Lina adds table of values and links it to coordinate system and to this node she also adds a third axis, the *z*-axis. She returns to linear function, which she had in the first map (and excluded in the second) as straight line and explains how it can be written as y=kx+m. She also explains the meaning of k and m. Another example of progressive differentiation is that she in addition to linear function also mentions other function classes as polynomial, rational, power, exponential and trigonometric functions, and so on. Williams (1998) noted that the experts in her study used a grouping that referred to classes of common types of functions, mentioning terms as exponential, polynomial, trigonometric and logarithmic. Thus here Lina's map has a feature that is typical for experts' maps.

Lina holds on to the nodes primitive function and integration and adds differentiated. One link seems to be not so well expressed: 'Functions can be solved graphically or....' It is not clear what she means here. It can be a mix up with solutions of equations but it can also be that she is thinking of problem solving with the aid of the graph of the function. This last proposition is an example of the student's lack of professional language, which many of the maps illustrate (Grevholm, 2004). While her second map has twelve nodes the third one has 25, more than twice as many. It strongly illustrates the progressive differentiation her function concept has undergone.

The maps were drawn over a period of 15 months where the student had no teaching of mathematics. But the maps show that great changes occur nevertheless. It seems as if the conceptual structure, the concept image, that the student is able to recall is getting richer as time goes by. One can of course argue that she is learning through repeated drawing of maps. This argument does not hold as can be seen by giving students the same mathematical problem again. There is normally no or little improvement in results even if the student has solved the problem once before. And the students did not keep the map that was once drawn and so could not rehearse it before drawing a new one.

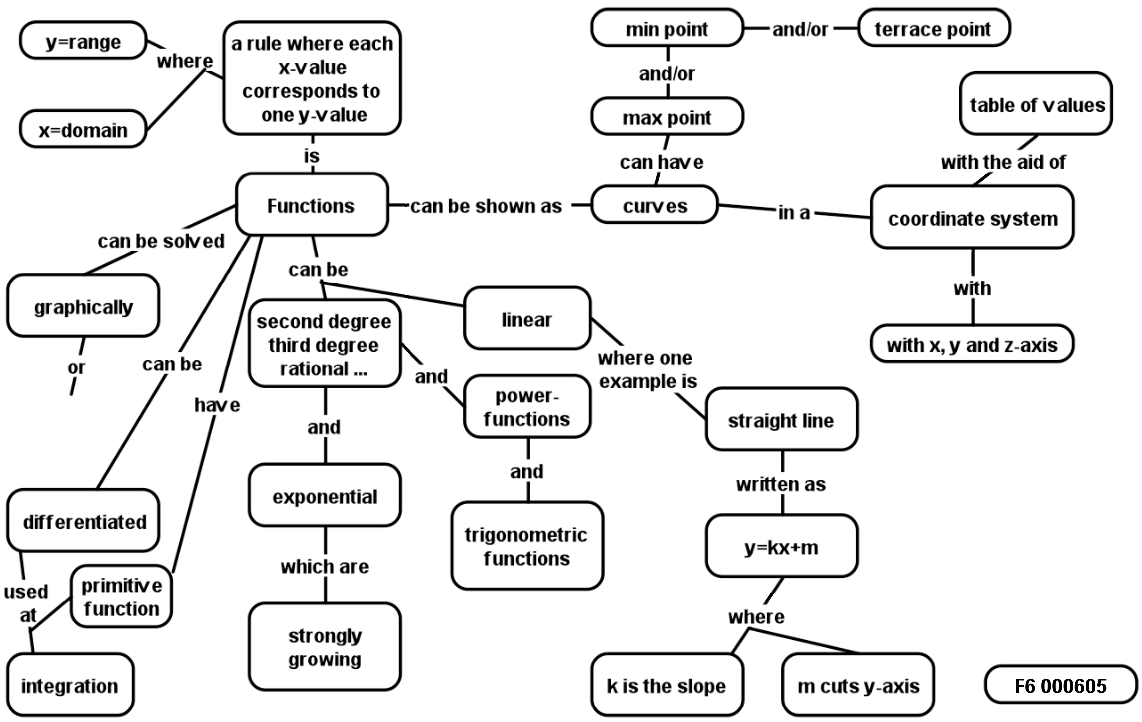

Figure 3. The third concept map of function drawn by Lina in June 2000.

The example of maps of one student given here is a typical one. The kind of development over time shown in the maps of Lina is not special in any way. The maps are very individual, each student has her way of drawing and is true to the model and design. The language in the maps reveals much about the student's ability to use the concepts involved in discussions (Grevholm, 2004). Lina's three concept maps illustrates Vollrath's claim (1994) that "the student reaches stages of understanding and that there is no final understanding". The conceptual structure undergoes changes over time and is dynamical and time-dependent. The maps indicate that we are dealing with a slow development and maybe as researchers we are sometimes too eager and do not wait for the concept development to take place and for the student to reach different stages?

6 Why are concepts maps rewarding as research tools?

What the researcher can learn about students' concept development from answers in a questionnaire and from the drawing of concept maps seems to be different. The verbal answers give short, often one-dimensional answers while the concept maps tends to give richer answers with more content and several dimensions of the concept. Students are vague and not enough specific when they try to explain verbally how they understand a concept.

The concept maps seem to reveal some properties of the concept development that are of interest. What are the advantages of using concept maps as answer form for students? From the example it is clear that the maps give the student better opportunities to express her concept image. It obviously invites to more multidimensional answers than a sentence which in its form is linear. The written answer does not open for hierarchy or additional lines of thought in the same way as the map. Knowledge that students express through a concept map seems to be lasting.

The way I used maps to make an a priori analysis of intended mathematical learning and then use another map to express students' answers in propositions and compare them has not been found in other mathematics education research reports (at least not in accessible journals). On the other hand, to study concept development over time maps have been used by several researchers. McGowen and Tall (1999) traced students cognitive development throughout a mathematics course by the use of concept maps at intervals during the course. They drew schematic diagrams of the maps of each student in order to see how students build maps by keeping some old elements, reorganising and introducing new elements. The results show that high achieving students "can show a level of flexible thinking building rich collages on anchoring concepts that develop in sophistication and power. The low achievers however reveal few stable concepts with cognitive collages that have few stable elements and leave the student with increasingly desperate efforts to use learned routines in inflexible and often inappropriate ways" (p. 287). Their findings are consistent with what I have shown. Thus it is obvious that concept maps can be used as a tool in research and in different ways as has been described here.

7 Concluding remarks

Obviously the problems that drive the research on student teachers' conceptual development derive from my experience as a mathematics teacher educator and originate from a desire to better understand the process during teacher education and to improve teacher education in mathematics. Can this be achieved if we know more about concept development? Can students experience more meaningful learning if we use new knowledge on concept development? Novak (1993) writes: "What remains to be demonstrated are the positive results that will occur in schools or other educational settings when the best that we know about human constructivism is applied widely. To my knowledge no school comes close to wide-scale use of such practices, even though there are no financial or human constraints that preclude this."

Can mathematics teacher educators design better learning situations for students when they know more about the cognitive development of students? Improvement of our knowledge on student teachers' development of concepts during the education might contribute in a constructive way to the redesign of teacher education. One outcome from the study related here was that the learning of the teacher educators resulted in a project for development of the professional language of a mathematics teacher, which was influential for both the students and the teacher educators (Grevholm, 2003b; Grevholm & Holmberg, 2004, Grevholm, 2004). Another outcome is a model of mathematics teacher education in the form of a concept map showing the development of a professional identity for the teacher (Grevholm, 2006, 2007).

References

Afamasaga-Fuatai, K. (1998). Learning to solve mathematics problems through concept mapping & Vee-mapping. Samoa: National University of Samoa.

Artigue, M. (2002). Didactic engineering and the complexity of learning processes in classroom situations. In C. Bergsten, G. Dahland, & B. Grevholm (Eds.), Research and action in the mathematics classroom. Proceedings of MADIF 2 (pp. 5-20). Linköping: SMDF.

Ausubel, D. (1963). The psychology of meaningful verbal learning. New York: Grune & Stratton.

Ausubel. D. (2001). The acquisition and retention of knowledge. A cognitive view. Dordrecht: Kluwer Academic Press.

Ausubel, D., Novak, J. D., Hanesian, H. (1978). *Educational psychology: A cognitive view*. New York: Holt, Rinehart and Winston.

Grevholm, B. (1998). Dokumentation av utveckling av matematikkursen Tal och algebra, 10 poäng, GMA031 i utbildningen till grundskollärare för årskurs 4-9, matematik och naturvetenskap. Kristianstad: Högskolan Kristianstad.

Grevholm, B. (1999). Teacher students' development of concepts in mathematics and mathematics education. In G. Brekke & T. Breiteg (Eds.), Theory into practice. Proceedings from NORMA98 (pp. 138-145). Kristiansand: Agder College.

Grevholm, B. (2000). Teacher education in transition. The case of Sweden. Paper presented at WGA7 in International Conference of Mathematics Education 9 in Makuhari, Tokyo, Japan July 2000. Kristianstad: Högskolan i Kristianstad.

Grevholm, B. (2002). Vi skriver y=x+5. Vad betyder det? SMDF Medlemsblad nr 5, 17-36.

Grevholm, B. (2003a). Student teachers' conceptions of equations and inequalities. In Proceedings from the Pre ICME10 conference in Växjö, May 11-12, 2003. Växjö: Växjö University.

(available at http://www.msi.vxu.se/picme10/L4GB.pdf)

Grevholm, B. (2003b). To develop the ability of teacher students to reason mathematically. Report on project number 026/00 to the Council for Higher Education from Department of Mathematics and Science, Kristianstad University. Kristianstad: Kristianstad University.

Grevholm, B. (2004). Mathematics worth knowing for a prospective teacher. In B. Clarke, D. Clarke, G. Emanuelsson. B. Johansson, F. K. Lester, D. V. Lambdin, A. Wallby, & K. Wallby (Eds.), International Perspectives on Learning and Teaching Mathematics (pp. 519-536). Göteborg: National Center for Mathematics Education.

Grevholm, B. (2005). Research on student teachers' learning in mathematics and mathematics education. In H. Fujita, Y. Hashimoto, B. R. Hodgson, P. Y. Lee, S. Lerman & T. Sawada (Eds). Proceedings of the Ninth International Congress on Mathematics Education (pp. 131-132 & CD). Dordrecht: Kluwer Academic Publishers.

Grevholm, B. (2006). Matematikdidaktikens möjligheter i en forskningsbaserad lärarutbildning. In S. Ongstad (Ed.), Fag og didaktikk i lærerutdanning. Kunnskap i grenseland. (pp. 183-206). Oslo: Universitetsforlaget.

Grevholm, B. (2007). Why is mathematics teacher education problematic? In A. Andzans, D. Bonka, G. Lace, (Eds.), VII International conference Teaching mathematics: Retrospective and perspectives. Proceedings, May 11-12, 2007, Riga, (pp. 79-88). Riga: Latvijas Universitate.

Grevholm, B. & Holmberg, I. (2004). To develop the ability of teacher students to reason mathematically. Final report on project number 026/00 to the Council for Higher Education from Department of Mathematics and Science, Kristianstad University. Kristianstad: Kristianstad University.

Hansson, Ö. (2006). Studying the views of preservice teachers on the concept of function. Doctoral thesis. Luleå: Luleå tekniska universitet.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics. An introductory analysis. In J. Hiebert (Ed.), Conceptual and procedural knowledge: the case of mathematics (pp 1-27). Hillsdale, NJ: Lawrence Erlbaum.

McGowen, M. & Tall, D. (1999). Concept maps and schematic diagrams as devices for documenting growth of mathematical knwoledge. In O. Zaslavsky (Ed.), Proceedings of the 23rd Conference of PME (Vol. 3, pp. 281-288). Haifa, Israel:Technion-Israel, Inst. of Techn.

Novak, J. D. (1985). Metalearning and metaknowledge strategies to help students learn how to learn. In L. West & A. Pines (Eds.), Cognitive structure and conceptual change (pp. 189-207). New York: Academic Press.

Novak, J. D. (1993). Human constructivism: A unification of psychological and epistemological phenomena in meaning making. International Journal of Personal Construct Psychology 6, 167-193.

Novak, J. D. (1998). Learning, creating and using knowledge. Mahwah, New Jersey: Lawrence Erlbaum.

Novak, J. D. & Gowin, D. B. (1984). Learning how to learn. New York: Cambridge University Press.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. Educational Studies in Mathematics 12 (2), 151-169.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics 22, 1-36.

Vollrath, J. (1994). Reflections on mathematical concepts as starting points for didactical thinking. In R. Biehler, R. W. Scholtz, R. Stræsser, & B. Winkelmann (Eds.), Didactics of mathematics as a scientific discipline (pp. 61-72). Dordrecht: Kluwer.

Williams, C. G. (1998). Using concept maps to assess conceptual knowledge of function. Journal for Research in Mathematics Education 25(4), 414-421.