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CONCEPT MAPS & VEE DIAGRAMS AS TOOLS FOR LEARNING NEW MATHEMATICS TOPICS

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Abstract. This paper presents the data for 6 students who participated in a study that investigated the use of the metacogntive tools of concept maps and vee diagrams in learning and solving problems for selected mathematics topics. The six students used the tools to learn about new mathematics topics. Initially, students struggled to understand their new topics and the tools. However, with independent research and progressively mapping their findings on concept maps and vee diagrams, and with critiques and feedback from others, students eventually developed enhanced and deeper understandings of their chosen topics.

1 Introduction

The study reported here is part of a series of concept map and vee diagram studies (mapping studies) conducted as part of an undergraduate research course at the National University of Samoa (NUS). The mapping studies conducted with different cohorts over a number of semesters were driven by the need to explore how students' understanding of mathematics could be improved beyond the algorithmic and procedural knowledge that they are equipped with after years of schooling. Students also have difficulties communicating and arguing mathematically (Richards, 1991; Schoenfeld, 1996), transferring and applying what they know in solving novel problems (Afamasaga-Fuata'i, 2003a, 2002). Thus, it was important for mathematics education that research be conducted to explore students' perceptions of guiding principles for problems they are able to solve. For example, having students identify the relevant conceptual bases for a given problem and its solution would reveal the existing state of students' mathematical understanding and perceptions of what constitutes relevant mathematical knowledge. The case study of Nat (one of the cohort of 7-students) is presented in the paper entitled "*An undergraduate student's understanding of differential equations through concept maps and vee diagrams*" included in these proceedings, is one of a student who chose a familiar topic to concept map and vee diagram. His initial concept map and initial vee diagrams of problems were basically procedural. However, over the semester and with critical comments during presentations his maps/diagrams evolved to ones that were more conceptual and theoretical. Through the use of maps/diagrams, he gained a more comprehensive, integrated, and differentiated conceptual understanding of ordinary differential equations.

The six students (rest of 7-student cohort) whose data is reported here selected mathematics topics they had not encountered before in their recent advanced mathematics courses. Hence the challenge was for them to learn about and develop an understanding for the new topic through the construction of concept maps and vee diagrams (maps/diagrams). They were also expected to present their work publicly to their peers in a group setting and to the researcher in a one-on-one consultative session. During the presentations, they were expected to demonstrate and communicate their understanding of the new topic clearly and succinctly so that the critics (peers and researcher) could make sense of it. Part of the newly established socio-cultural practices in the classroom setting (socio-mathematical norms) were the expectations that they undertake independent research on their topic, be prepared to justify their constructions, address concerns raised and negotiate meanings during critiques.

The theoretical framework of the study is the Ausubel-Novak theory of meaningful learning, which describes meaningful learning as the process in which the student chooses to relate new information to existing knowledge (Ausubel, Novak & Hanesian, 1978; Novak, 2002, 1998, 1985). This process may be facilitated through the construction of concept maps and vee diagrams. Having students identify main concepts and organize them into a concept hierarchy of interconnecting nodes with propositional links can indicate the existing state of students' cognitive structures or patterns of meanings. The establishment of socio-mathematical norms of presenting individual work and critiquing peers' work is based on the principles of social constructivist and socio-linguistic perspectives which view the process of learning as being influenced and modulated by the nature of interactions and linguistic discourse undertaken in a social setting (Ball, 1993; Schoenfeld, 1996; Ernest, 1999; Richards, 1991; Knuth & Peressini, 2001).

2 Methodology & Data Analysis

The study was conducted as an exploratory teaching experiment (Steffe & D'Ambrosio, 1996) using the metacognitive tools of concept maps and vee diagrams (Novak, 1985, 2002) with students presenting their work for group and one-on-one critiques. After completing practice sessions in constructing maps/diagrams and presenting work in a social setting for critique, students selected their new topics for the application of the meta-cognitive tools. Students engaged in the cyclic process of presenting→ revising→ critiquing→ presenting for at least three iterations over the semester. The data from the six students consisted of progressive concept maps (4 versions) and vee diagrams of four problems (at least 2 versions each). The six students (Student 1 to 6) chose the topics Laplace's transform, trigonometric approximations, least squares polynomial approximations, multivariable functions and their derivatives, partial differential equations and numerical methods of solving first order differential equations. Each student's case is presented first, beginning with the concept map data and then followed by a general discussion of their vee diagram data, before discussion of general themes.

2.1 Concept Map Analysis

The qualitative approach adopted in the analysis of the data is a modification of the Novak scheme of scoring concept maps (Novak & Gowin, 1984). This paper uses only counts of occurrences of each criterion. Collectively, the criteria assess students' concept maps in terms of the *structural complexity of the network* of concepts, *nature of the contents* (entries) of concept boxes (nodes) and *valid propositions.* The structural criteria indicate the extent of integrative crosslinks between concepts and progressive differentiation between levels whilst the contents criteria indicate the nature of students' perceptions of mathematical knowledge. Valid propositions are those formed when 2 valid nodes are interconnected with appropriate linking words correctly describing the nature of the inter-relationship. The structural criteria are also assessed in terms of average hierarchical levels per sub-branch, multiple branching nodes, sub-branches and main branches. Particular examples are those used to illustrate concepts. Inappropriate entries at nodes are those that describe a procedural step, redundant concepts and linking-word-type. Redundant entries indicate students' tendency to learn information as isolated from each other instead of identifying potential integrative crosslinks with the first occurrence of the concept or consider a re-organization of the concept hierarchy. Linking-word-type indicates students' difficulties to distinguish between a "mathematical concept" and descriptive phrases. A proposition is invalid if linking words were missing, incorrect or end nodes had inappropriate entries. Data for the six students' progressive concept maps (first and final maps) are shown in Tables 1 and 2 below.

Reference	**Art**	**Art**	**Ada**	**Ada**	**Lou**	**Lou**	**Asi**	**Asi**	**Afa**	**Afa**	**Les**	**Les**
Student	**1**		**2**		**3**		**4**		**5**		**6**	
Map	**1**	**4**	**1**	**4**	**1**	**4**	**1**	**4**	**1**	**4**	**1**	**4**
Concepts	14	24	8	19	13	43	12	51	13	43	36	84
Examples	0	0	2	0	0	0	1	0	2	2	2	4
Definitional	0	2	2	6	0	0	3	0	0	0	0	0
Inappropriate	3	0	2	2	0	0	1	0	0	1	2	17
Total	17	26	14	27	13	43	17	51	15	46	40	105
Concepts	82%	92%	57%	70%	100%	100%	71%	100%	87%	93%	90%	80%
Examples	0%	0%	14%	0%	0%	0%	6%	0%	13%	4%	5%	4%
Definitional	0%	8%	14%	22%	0%	0%	18%	0%	0%	0%	0%	0%
Inappropriate	18%	0%	14%	7%	0%	0%	6%	0%	0%	2%	5%	16%
Valid Prop	6	18	6	17	12	41	6	42	14	47	40	87
Invalid Prop	11	5	8	14	0	1	10	17	2	2	5	28
% Valid	35%	78%	43%	55%	100%	98%	38%	71%	88%	96%	89%	76%
Integrative Crosslinks	1	1	0	0	4	6	0	4	0	4	7	18

Table 1: Contents, Propositions and Crosslinks Criteria

Student 1's topic was Laplace's transform (LT). From his research, Art selected a few concepts for his first map to provide a definition for LT, and to illustrate how they are used in solving initial value problems. His first concept map had 17 nodes of which 14 were valid with 3 inappropriate ones due to procedural, redundant and link-word-type entries. Only 35% of the propositions were valid with only one integrative crosslink, see Table 1. The high proportion of invalid propositions was due to missing or inappropriate linking words. At the first group critique, critical comments focussed on the need to reconsider the hierarchical order of concepts, missing relevant concepts and inappropriate concept labels. Comments from subsequent critiques over the semester pinpointed areas of confusion, which guided Art to sections of his map that needed re-organization and re-structuring to enhance its intended meaning. By the end of semester, Art's final concept map showed an increase in valid nodes (from 14 to 24) with significantly more valid propositions (from 35% to 78%), more sub-branches (from 6 to 10), higher average hierarchical levels per sub-branch (from 4 to 6), an additional main branch (from 3 to 4) and an increased number of multiple branching nodes (from 4 to 8), see Table 2. Overall, Art's final concept map had become more integrated and complex as his understanding expanded and became more enriched as a result of critiques, revisions and individual research. For example, he wrote: "*with concept maps, its uses that I have experienced from the semester is that they broaden my understanding of my chosen topic... (constructing concept maps) allows the writer to easily understand his own topic through substantial and more comprehensive links and to simply make changes from comments in class presentations.*"

Reference	**Art**	**Art**	**Ada**	**Ada**	**Lou**	**Lou**	**Asi**	**Asi**	**Afa**	**Afa**	**Les**	**Les**
Student	**1**		**2**		**3**		**4**		**5**		**6**	
Map	**1**	**4**	**1**	**4**	**1**	**4**	**1**	**4**	**1**	**4**	**1**	**4**
Sub-Branches	6	10	4	8	3	15	6	20	4	19	14	32
Hierarchical Levels	4	6	4	7	8	8	4	7	4	9	10	15
Main Branches	3	4	2	3	1	6	3	4	3	5	4	5
Multiple Branching Nodes at:												
Level 1	2	2				2	2	2	2			
Level 2	2	3,2	2			3		3,2				
Level 3	2,2	2,2		2		2,2	2,3	2,6		4	2	2
Level 4		2	2	5	2	2		3,2,3	2	2,4,2		
Level 5			2			2,2,2,3	2	3		2	2	2
Level 6		2			2	2,2				2	2,2	2
Level 7		2				2,3		2,2,2	2	2	2,2	2
Level 8						2		3		2	3,2	
Level 9										2		2,3
Level 10				2						2		2,2
Level 11										2	2	2
Level 12										2,2	2	3,2,3,2
Level 13										2		2
Level 14										2		2,2
Total # Multiple Branching Nodes	4	8	3	3	2	14	4	13	3	15	10	16

Table 2: Structural Criteria

Student 2's topic was Trigonometric Approximations. Ada found his topic hard, but after reading a few textbooks, he chose to approach his topic using his background knowledge of Taylor's polynomial. For example, he chose to demonstrate the concept of approximations of values of a compound trigonometric function by successively approaching the point. Thus, the first map was mainly procedural but with time and critiques, his final map evolved into a more conceptual one with the demonstration of method of application relegated to a vee diagram. For example, Ada wrote in his report: "*I was forced to look for key concepts involved in Taylor's polynomial and how they are interrelated to other branches of mathematics. I sought how the terms in the series functioned and what relationship they had to practical applications like speed, acceleration and distance, forming the ability to use this tool in other situations. … Overall, it was a difficult but helpful experience in which I have a deeper understanding of Taylor's polynomial but as yet many unanswered questions.*" Ada's final concept map had relatively more valid concepts (from 8 to 19). Unfortunately, 6 nodes had definitional phrases, which require further analysis to form more succinct conceptual entries. There was also an increase in valid propositions (from 6 to 17) but the inordinately high invalid propositions (from 8 to 14) is due to missing linking words, and inappropriate-end-nodes (definitional phrases).

Student 3's topic was least squares polynomial approximation (LSPA). Lou's first concept map consisted of one main branch with only two multiple branching nodes, and four integrative crosslinks, see Table 2 for data and the left map in Figure 1. From the first group critique, she realized that her map did not provide sufficient concepts to explain the main ideas relevant to her topic particularly the concept of *errors* in spite of having included the concept of *squared differences*. Hence, with more readings, and research, she added in concepts of *errors, five-point-least-square-polynomial, smoothing formula, data smoothing* and *n^{th} degree* to name a few, for her first revisions. However, as she wrote in her report: "*Despite the clustered and plentiful information given in my map, the main concept of errors is lost. This is because there was less emphasis on understanding the topic. Rather, a collection of various concepts seemed more important at the time. Hence, an improved map would require meaningful concepts, mathematical formula, neater presentation, and simple examples. I learnt here that the basic idea behind the topic is that there is an error and everything falls around the minimising of this error.*" With more critiques, further research for additional concepts and subsequent revisions, Lou realized that the concepts of *Least-square polynomial P(x), Function F(x)* and *Error = Y(x) – P(x)* have to be positioned appropriately and the case for *continuous data* required further clarification. By her third revision, Lou noted that her revised map "*showed a clear hierarchy of linking concepts … hence it was easier to follow what the map is trying to tell us. However, there is still work to be done on clarification, organization and available information.*" She also learnt that "*organization plays a huge role in making the map comprehensive.*" With more critiques and revisions, Lou's final concept turned out to be a "*a much more effective one in terms of understanding the concepts related to the topic (LSPA). So, the idea of errors was clear, its application and determination was also specified, and the table for clarification of Newton's formula, was also a great improvement.* " In summing up her experiences in the study, she wrote: "*I have now seen an evolvement from a very basic map to a more complicated one. The surprising fact discovered is that the basic map (i.e. first map) was more confusing than the resulting one (i.e. final map).*" This is quite a revealing statement about the value of her final map as a more meaningful, comprehensive and informative piece of work. Part of Lou's final map is shown in the right map in Figure 1 for comparison to her first attempt.

Student 4's topic was multivariable functions and their derivatives. Asi's first concept map had 12 valid nodes with 4 invalid nodes due to a definitional phrase and inappropriate entries. The invalid propositions (10 out of 16) were due to missing linking words or inappropriate nodes. In spite of Asi's efforts, the group found her first concept map presentation confusing due to vague and inappropriate linking words. Asi then revised and reorganized her concept hierarchy to make the map more meaningful. Subsequent one-on-one and group critiques over the semester eventually resulted in a final map which was more differentiated with increased multiple branching nodes (from 4 to 13), and sub-branches (from 6 to 20) with a higher average hierarchical levels per sub-branch (from 4 to 7). In response to critical comments, Asi reorganized the concept hierarchy, revised linking words to make them more descriptive of interconnections, created more sub-branches, and provided meaningful integrative crosslinks to improve the clarity and organization of information. Overall, the final map had significantly more valid nodes (from 12 to 51) and valid propositions (from 6 to 42). Asi wrote in her final report: "*To me, using concept maps has given me a chance to learn more of my research topic.*"

Student 5's topic was numerical methods of solving first order differential equations. Afa's first concept map had 15 valid nodes of which 2 were examples, 3 multiple branching nodes, and 4 sub-branches with average hierarchical levels of 4 per sub-branch. Through critiques and subsequent revisions, his final map evolved into one that was more differentiated and enriching with substantial increases in sub-branches (from 4 to 19), average hierarchical levels per sub-branch (from 4 to 9), main branches (from 3 to 5), integrative crosslinks (from 0 to 4), and multiple branching nodes (from 3 to 15). Overall, valid propositions increased from 14 to 47.

Student 6's topic was partial differential equations (pdes). Les' first concept map had 38 valid nodes with only 2 invalid ones due to redundant entries. The map differentiated between first and second order pdes with further differentiation at lower levels into homogenous and non-homogeneous types, and had 40 valid propositions with only 5 invalid ones due to incorrect/vague linking words and inappropriate end nodes. With further critiques and subsequent revisions, Les' final map eventually evolved into one that was substantially more complex with increases in sub-branches (from 14 to 32), average hierarchical levels per sub-branch (from 10 to 15), multiple branching nodes (from 10 to 16) and integrative crosslinks (from 7 to 18). Valid propositions had also increased from 40 to 87. However, the higher number of invalid propositions (from 5 to 28) is due to an increased number of inappropriate nodes due to procedural, redundant and linking-word-type entries and missing linking words. Les created additional sub-branches in the final concept map to provide conceptual bases for his vee diagram problems. He wrote in his final report: "*I myself understand fully the path from one concept to another and how a conclusion can be obtained because I created the concept maps.*" He continued on to state that "*From my experience in laying out my concept map I have learnt that differentiating first order and its special cases and second order and its special cases avoids confusion. It helps me to classify each pde I come across so that I could see the big picture.* "

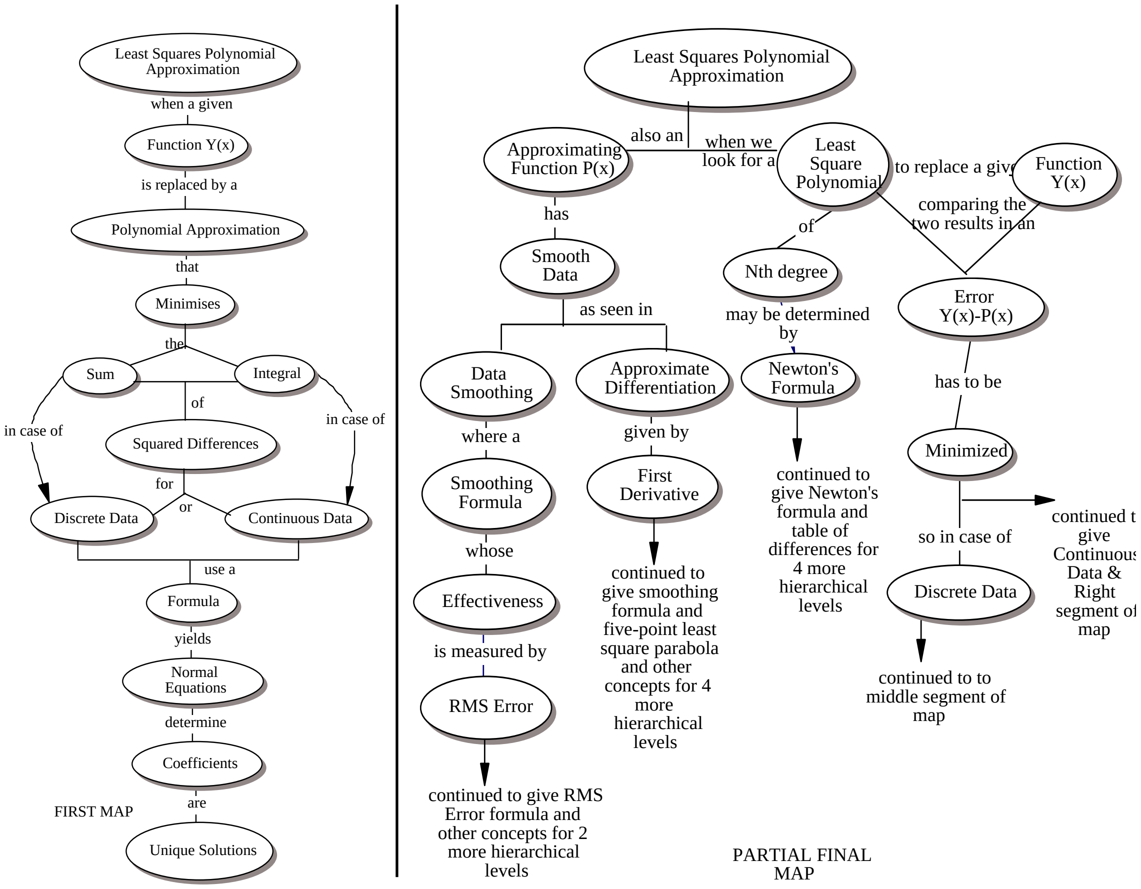

Figure 1. Lou's first concept map & partial final concept map.

2.2 Vee Map Analysis

The structure of the vee diagram (see Figure 2) with its various labels and guiding questions provide a systematic guide to students to reason from the problem context (Event/Object) and given information (Records) in identifying relevant principles, theorems, formal definitions and major rules (Principles) and (Concepts) which can guide the development of appropriate methods and procedures (Transformations) to find an answer (Knowledge Claim) to the (Focus Question). The arrow indicates that there is a continuous interplay between the two sides as students reason through the various sections of the vee. Vee diagrams are qualitatively analysed to determine whether or not the conceptual and methodological sides mutually support each other. That is, do the listed principles support the given solution? Are the listed principles the most relevant for given solution? Is the knowledge claim supported by the listed principles and transformations? As Gowin (1981) points out: "*The structure of knowledge may be characterized (in any field) by its telling questions, key concepts and conceptual systems; by its reliable methods and techniques of work…*" (pp. 87-88).

Therefore, in this study, the vee diagram is used as a tool to not only asses students' proficiency in solving a problem but also the depth and extent of the conceptual bases of this proficiency requiring students to identify the mathematical principles and concepts underlying listed methods and procedures. To these ends, students' vee diagrams are assessed qualitatively in terms of one overall criteria and a more specific one. Specifically, the overall criteria assesses the appropriateness of entries in each section according to the guiding questions in Figure 2 and the given problem whilst the specific criteria refers to the extent of integration and correspondence between listed principles and listed main steps. The focus is on the relevance, appropriateness and completeness of listed principles in relation to methods and procedures listed under Transformations.

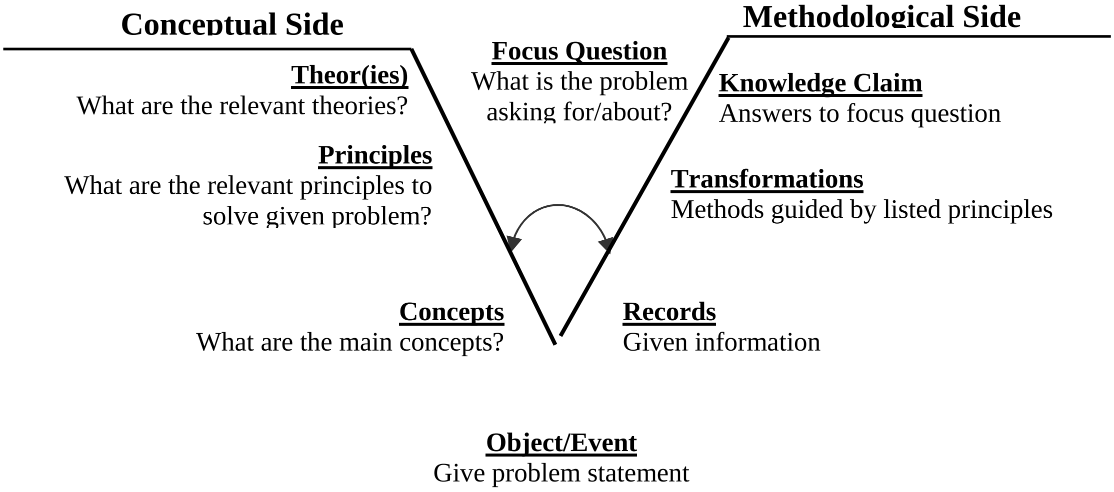

Figure 2. Problem solving vee diagram of mathematics problems (Afamasaga-Fuata'i (1998) modified from Novak & Gowin, 1984, Gowin, 1981)

In consideration of page constraints, instead of presenting each student's vee diagrams, general themes emanating from their work are presented and discussed. For example, in terms of the overall criteria, all students had satisfactory entries for the sections "Theories", "Concepts", "Records", and "Knowledge Claim" as these were basically extracted and inferred from the problem statements. Also because they were free to select their problems, obtaining the correct answers was not problematic. However, what caused a lot of critical comments and numerous revisions were the inappropriate entries for the sections on "Principles", and "Transformations." The general weakness with the former is the language used to describe principles. The intention is clear but wording were initially too procedural in contrast to theoretical statements of general rules and formal definitions. There was a tendency to provide only formulas without clarifications subsequently leading to ambiguities. With transformations, listed main steps did not always have supporting principles on the vee. In terms of the specific criteria, most of the students scored low in their initial maps. However, with critiques, evolving comprehensive concept maps, and subsequent revisions over the semester, students' listed principles improved to become more conceptual statements, and the lists expanded to include relevant principles to support listed main steps in the transformation section. As one of the students wrote in the final report: "*the principles section required much thought and reorganising … my struggle was to ensure the principles were general statements and formula that became tools for solving the given problem.*"

3 Discussion

Learning a new topic and learning to use concept maps and vee diagrams were big demands of students as Lou puts it: "I began my semester of reading a page over and over again, looking at examples and reading the same page one more time, only to realise that I had to reorganize my concepts again. This became my routine for the study of concept mapping: reading, checking, writing, organizing,.., reading, checking, and onwards I went. " However, by the end of the semester, Lou wrote: "it would have been impossible to reach a more comprehensive map without the input from the class and lecturer." All six students found that to construct a map that made sense to the critics, they had to research more, continually revise and re-organize the concept hierarchies. Furthermore, the construction of the vee diagrams was greatly facilitated when based on a comprehensive integrated and differentiated concept map as evidenced by the creation of additional branches on concept maps to illustrate guiding principles for a method on a vee diagram. In doing these activities, students learnt more about the conceptual structure of their topics in more meaningful ways and at a deeper level as well becoming proficient with the relevant methods and procedures. As Lou sums up her experiences: "When I presented my last concept map to class, it dawned on me that I had finally understood what I was struggling to know since the beginning of semester. The words, 'Least square polynomial approximations' no longer threatened me. I could close my eyes and summarize this topic to someone else without a doubt in my head that what I would be saying made sense."

The concurrent use of the two tools in learning about a new topic contributed significantly in highlighting the close correspondence between the conceptual structure of a mathematics topic and its methods. For example, a student wrote: "*With the help of constructive comments from critiques, I was able to work on appropriate vee* *maps that elaborated on the concept map. This was the fundamental role of the vee diagrams – to elaborate on the concepts shown by the concept maps. With this elaboration, I was able to understand the topic even better.*" That is, possessing only a procedural and algorithmic view of mathematics is limiting. Instead, an enriched knowledge of the conceptual bases of methods, and in-depth knowledge of the conceptual structure can motivate students to learn more about their topic. For example, Lou wrote: "*Making sense out of a difficult topic through concept mapping was the miracle that I was enlightened with. In addition to this awesome discovery, I realised that the miracle was endless. That is, I could go on learning more about least square polynomial approximations because there is always more concepts waiting to be discovered, analysed and revised. So concept mapping is also a tool for extending one's knowledge."*

4 Conclusions

Students' progressive concept maps and vee diagrams showed improvement over time as a consequence of group presentations, individual work, peer critique and one-on-one consultations. That is, students' concept maps had evolved over the semester into maps that were more meaningfully integrated and differentiated and more enriching in its conceptual structure. Their vee diagrams showed growth in their correspondence between methods of solutions and listed principles and enhancement of the conceptual integrity of identified principles. The increased structural and conceptual complexity reflected the growth in the extent and depth of students' understanding of the links between theoretical principles and methods of solutions.

The established socio-mathematical norms of critiques and presentations contributed significantly to the developing quality and refinement of students' evolving understanding of their topics. The act of talking aloud (presenting and justifying to peers) required a level of reflection that aided in the problem solving process. Talking aloud has the power to change students' performance (Richards, 1991, p.37) as evident in the evolving maps/diagrams.

One of the value claims from students' perspectives is the self-realization that the construction of maps/diagrams requires and demands a much deeper understanding of interconnections than simply knowing what the main concepts and formulas are. Although time consuming, the construction of maps/diagrams facilitates learning the structure of a topic in more meaningful ways. Furthermore, students realized that the communication of their understanding is more effective if concepts are arranged in a hierarchical order complete with appropriate labels, meaningful links with concise and suitable linking words. Another value claim of the study is the potential of applying the metacognitive tools to other subject areas by the same students. This is succinctly captured by Lou's comments in her final report: "*I was able to apply the theory of concept mapping to my other subjects and found that I became relaxed when confronted with a difficult topic. Then I was rewarded with good marks. Before I learnt of concept maps, my initial response to a difficult subject would be to panic. Then I would try to break the problem down, read, research, memorize, and do all the things an average student does before understanding some of the topic being studied. Now I wish that our high school teachers had taught us about concept mapping. It would have done wonders for me.*"

There are still problematic areas that need attention mainly due to the newness of the tool which students need to overcome with more practice and more time. As one of the students noted, collecting a list of relevant concepts and formulas is one thing but actually figuring out how they should all be interconnected is another. That is the task of determining the most appropriate linking words to concisely describe the nature of the interconnection still requires further improvement. From this study, the 6 students appreciated the utility of the maps/diagrams as means of illustrating conceptual interconnections within a topic and highlighting connection between principles and procedural steps. Students also appreciated the value of the tools in mapping their growing understanding and as means of communicating that understanding to others in a social setting. Findings from this cohort suggest that concept maps and vee diagrams are potentially viable tools for developing a deeper understanding of the structure of mathematics.

5 References

Afamasaga-Fuata'i, K. (2003a). *Numeracy in Samoa: From Trends & Concerns to Strategies.* Paper presented at the Samoa Principals Conference, Department of Education, EFKS Hall, Samoa, January 28-30, 2003.

Afamasaga-Fuata'i, K. (2003b). *Mathematics Education: Is it heading forward or backward?* Paper presented at the Measina Conference, Institute of Samoan Studies, National University of Samoa. December 4, 2003.

Afamasaga-Fuata'i, K. (2002a). *Vee diagrams & concept maps in mathematics problem solving.* Paper presented at the Pacific Education Conference (PEC 2002), Department of Education, American Samoa, July 23, 2002.

Afamasaga-Fuata'i, K. (2002b). *A Samoan perspective on Pacific mathematics education.* Keynote Address. Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia (MERGA-25), July 7-10, 2002 (pp. 1-13), University of Auckland, New Zealand.

Afamasaga-Fuata'i, K. (2001). *Enhancing students' understanding of mathematics using concept maps & vee diagrams.* Paper presented at the International Conference on Mathematics Education (ICME), Northeast Normal University of China, Changchun, China, August 16-22, 2001.

Afamasaga-Fuata'i, K. (1998). Learning to Solve Mathematics Problems Through Concept Mapping & Vee Mapping. National University of Samoa.

Ausubel, D. P., Novak, J. D., & Hanesian, H. (1978). *Educational Psychology: A Cognitive View*. New York: Holt, Rhinehart and Winston. Reprinted 1986, New York: Werbel & Peck.

Ball, D. (1993). With an eye on the mathematical horizon: dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, 93(4), 373-397.

Ernest, P. (1999). Forms of knowledge in mathematics and mathematics education: philosophical and rhetorical perspectives. *Educational Studies in Mathematics*, 38, 67-83.

Gowin, D. B. (1981). *Educating*. Ithaca, NY: Cornell University Press.

Knuth, E., & Peressini, D. (2001). A theoretical framework for examining discourse in mathematics classrooms. *Focus on Learning Problems in Mathematics*, 23(2 & 3), 5-22.

Novak, J. (2002). Meaningful learning: the essential factor for conceptual change in limited or appropriate propositional hierarchies (LIPHs) leading to empowerment of learners. *Science Education*, 86(4), 548-571.

Novak, J. D. (1998). Learning, Creating, and Using knowledge: Concept Maps as Facilitative Tools in Schools and Corporations. Mahwah, NJ: Academic Press.

Novak, J. D. (1985). Metalearning and metaknowledge strategies to help students learn how to learn. In L. H. West, & A. L. Pines (eds.) *Cognitive Structure and Conceptual Change* (pp. 189-209). Orlando, FL: Academic Press.

Novak, J. D. & Gowin, D. B. (1984). *Learning How to Learn*. Cambridge University Press.

Richards, J. (1991). Mathematical discussions. In E. von Glaserfeld, (ed.), *Radical constructivism in mathematics education* (pp. 13-51). London: Kluwer Academic Publishers.

Schoenfeld, A. H. (1996). In fostering communities of inquiry, must it matter that the teacher knows "the answer." *For the Learning of Mathematics*, 16(3), 569-600.

Steffe, L. P., & D'Ambrosio, B. S. (1996). Using teaching experiments to enhance understanding of students' mathematics. In D. F. Treagust, R. Duit, & B. F. Fraser (eds.), *Improving teaching and learning in science and mathematics* (pp. 65-76). Teachers College Press, Columbia University, New York.