Concept Maps: Theory, Methodology, Technology
Proc. of the First Int. Conference on Concept Mapping
A. J. Cañas, J. D. Novak, F. M. González, Eds.
Pamplona, Spain 2004

AN UNDERGRADUATE STUDENT'S UNDERSTANDING OF DIFFERENTIAL EQUATIONS THROUGH CONCEPT MAPS AND VEE DIAGRAMS

Karoline Afamasaga-Fuata'i, University of New England
Email: kafamasa@pobox.une.edu.au

Abstract. The paper presents the case of a student (Nat) who participated in a semester long study which investigated the impact of using concept maps and vee diagrams on students' understanding of advanced mathematics topics. Through the construction of concept maps and vee diagrams, Nat realized that there was a need for him to deeply reflect on *what* he really knows, determine *how* to use what he knows, identify *when* to use *which* knowledge, and be able to justify *why* using valid mathematical arguments. He found that simply knowing formal definitions and mathematical principles verbatim did not necessarily guarantee an in-depth understanding of the complexity of inter-connections and inter-linkages between mathematical concepts and procedures. During seminar presentations and one-on-one consultations, Nat found that using the constructed concept maps and vee diagrams greatly facilitated discussions, critiques, dialogues and communication. The paper discusses the results from this case study and some implications for teaching mathematics.

1 Introduction

The use of concept maps and vee diagrams in learning subject matter more meaningfully and more effectively has been the focus of numerous researches (Mintzes, Wandersee & Novak, 1998; Novak, 2002, 1998; Williams, 1998; Liyanage & Thomas, 2002). The author had also studied the use of these meta-cognitive tools in mathematics problem solving by secondary students (Afamasaga-Fuata'i, 1999, 1998) and undergraduate students (Afamasaga-Fuata'i, 2002a, 2001, 2000, 1999). The undergraduate mapping studies (concept map and vee diagram studies) evolved out of a need to seek innovative ways in which Samoan students' mathematics learning can be improved beyond their technical proficiency in applying known procedures and algorithms (Afamasaga-Fuata'i, 2003a, 2003b, 2002b). However, before presenting Nat's data, the following sections provide an overview, conceptual framework, and methodology of the mapping studies. Data analysis is suggested by Ausubel-Novak's theory of meaningful learning (Ausubel, Novak & Hanesian, 1978; Novak & Gowin, 1984), social constructivistic and socio-linguistic perspectives. A discussion of possible influences of the tools and socio-mathematical norms on Nat's progressive understanding of, and efficiency in solving differential equations (D.E.) problems will be followed by recommendations for teaching mathematics.

The mapping studies explored the impact constructing concept maps and vee diagrams (maps/diagrams) has on students' mathematical understanding particularly their fluency with the language of mathematics and efficiency in communicating in a classroom community of mathematics students including the researcher. Ways in which students' mathematical perceptions are influenced as a result of constructing maps/diagrams and publicly communicating that understanding were also examined. Subsequently, new socio-mathematical norms were established to support the communication of students' evolving understanding bearing in mind that these students have been through years of predominantly traditional mathematics learning where discourse is mainly to transmit knowledge and mathematical discussions (inquiry mathematics) recommended by Richards (1991) is not the norm up to this point (Afamasaga-Fuata'i, 2003a, 2003b, 2002b).

2 Theoretical Perspectives

To explain the processes of knowledge construction and meaningful learning, Ausubel-Novak's cognitive theory of meaningful learning guides the study particularly its principles of assimilation and integration of new and old knowledge into existing knowledge structures. The deliberate linking of concepts to relevant existing concepts may be by progressive differentiation and/or integrative reconciliation. Figure 1 shows a partial view of Nat's first concept map in which the concept "$\mathbf{a_0\ dy/dx + a_1\ y = Q(x)}$" is progressively differentiated into two links to connect to concepts "$\mathbf{Q(x) = 0}$" and "$\mathbf{Q(x) \neq 0}$." In contrast, the two concepts "**Function (x, y)**" and "**Integrating Factor**" are integrated and reconciled with the less general concept "**General Solution**." According to this theory, students' cognitive structure should be hierarchically organized with more inclusive, more general concepts and propositions superordinate to less inclusive, more specific concepts and propositions to facilitate assimilation and retention of new knowledge (Ausubel et. al, 1978, Novak, 2002, 1998).

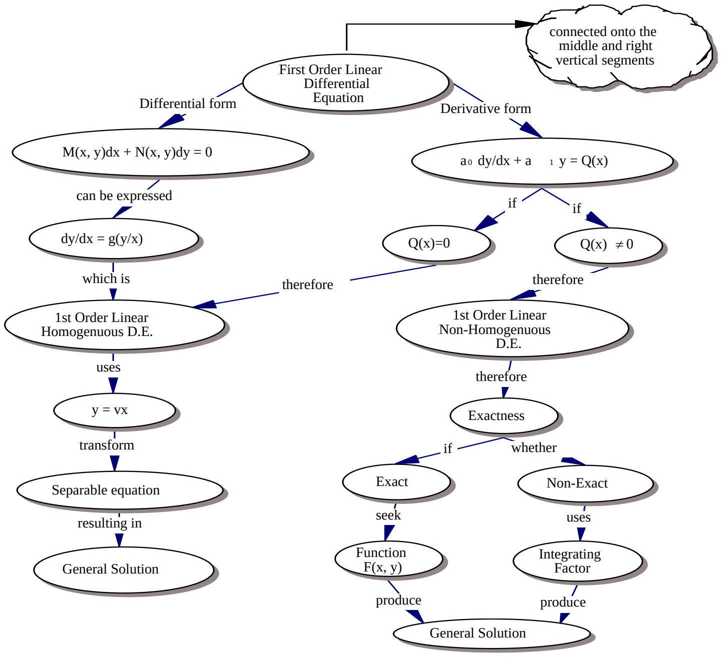

Figure 1. Left vertical segment. Nat's first concept map

The social constructivist perspective provides the opportunity for the development of students' metacognitive skills within a classroom setting particularly as they actively construct mathematical knowledge and reflect on their thinking. Through public presentations, students interact with others as they seek to negotiate meaning, justify and argue the validity of their work. Collectively the cognitive (Ausubel et. al, 1978) and socio-linguistic views (Richards, 1991; Knuth & Peressini, 2001) highlight that "teaching and learning mathematics involves being initiated into mathematical ways of knowing, ideas and practices of the mathematical community and making these ideas and practices meaningful at an individual level" (Ernest, 1999) as well as having the ability to maintain and conduct mathematical discussions (Richards, 1991).

3 Methodology & Data Analysis

The methodology was a qualitative, exploratory teaching experiment (Steffe & D'Ambrosio, 1996) conducted over a semester of 14 weeks with different cohorts to introduce students to the meta-cognitive tools (Novak, 1985). Epistemological principles of building upon prior knowledge, group work, negotiation of meanings, consensus and provision of time-in-class to allow students to reflect on their own understanding guided activities. For example, a familiarization phase introduced students to the new tools and socio-mathematical norms of group and one-on-one (1:1) critiques, including the expectation that students address critical comments from peers and researcher, and then later on critique peers' presented work. Students underwent 3 cycles of group and 1:1 critiques before completing a final report. Topics selected by Nat's cohort of 7 students included partial differential equations, approximation methods, multiple variable functions, Laplace's transform, least square polynomial, and trigonometric approximations. This paper reports the data from Nat's D.E. maps/diagrams. The data for the rest of the 7 students is presented in a second paper for this conference entitled "*Concept maps & vee diagrams as tools for learning new mathematics topics*." Data collected consisted of Nat's progressive concept maps (4 versions) and progressive vee diagrams of 4 problems (at least 2 versions each), and final reports. Students' perceptions of the value of maps/diagrams were obtained through written responses to questions on the advantages/disadvantages of using the maps/diagrams to learn mathematics. In the following sections, Nat's concept map data is discussed first followed by those for the vee diagrams.

3.2 Concept Map Data Analysis

The literature points to different ways of assessing and scoring concept maps (Novak & Gowin, 1984; Ruiz-Primo & Shavelson, 1996; Liyanage & Thomas, 2002) however for this study, a qualitative approach is adopted. Students' concept maps are assessed mainly, using counts, in terms of the *complexity of the network structure* of concepts, *nature of the contents* (entries) within concept boxes (nodes) and *valid propositions,* see Table 1. The structural criteria indicate the depth of differentiation and extent of integration between concepts whilst the contents criteria reflect the nature of students' perceptions of mathematical knowledge. Valid propositions are formed by connecting valid nodes with suitable linking words (correctly describing the nature of interrelationship); that is, valid (node→linking words→node) triads.

TOPIC: Differential Equations	First Map	Final Map	%Increase
CRITERIA: CONTENTS	**Count**	**Count**	
A: Conceptual Contents - TOTAL A	**45 (86.5%)**	**72 (88.9%)**	**2.4**
Concept Names/Labels	31	27	-13
Concept Symbols/Expressions	4	13	225
Symbols	0	4	
Mathematical Statements/Expressions	8	13	63
Names of Methods	2	6	200
General Formulas/Expressions	0	8	800
Formula Concepts	0	1	100
B: Inappropriate Entries - TOTAL B	**7 (13.5%)**	**5 (6.2%)**	**-7.3**
Procedures	0	1	
Linking Words used in concept boxes	0	2	
Redundant Entries	7	2	
C: Definitional Entries – Parts of definitions - **TOTAL C**	**0**	**3**	
D: Examples - TOTAL D	**0**	**1**	
TOTALS A+B+C+D	**52**	**81**	**5 5.8**
CRITERIA: PROPOSITIONS			
Valid Propositions	33	54	63.6
Invalid Propositions	16	19	18.8
CRITERIA: STRUCTURAL			
Main Branches	8	8	0
Sub-branches	13	17	31
Average Hierarchical Levels per Sub-branch	7	11	57
Integrative Crosslinks - Between (sub-)branches at same level	2	2	
Integrative Crosslinks - Between (sub-)branches at different levels	1	12	
TOTAL CROSSLINKS	3	14	366
Multiple Branching - Progressive Differentiation Links from Nodes at:			
Level 2	5	2	
Level 3	2	3	
Level 4	2,2,2	3	
Level 5		5,2,2	
Level 6	2,2	2,2	
Level 7	2,2	7	
Level 8		2	
Level 9		2	
Level 10		2,3	
Level 14		2	
Total Number of Nodes with Multiple Branching	9	14	56

Table 1: Contents, Propositions and Structural Criteria

The occurrences of lengthy statements (*definitional phrases*) require further analysis to identify concepts. Other types, categorized as *inappropriate entries* such as *procedural* entries are more appropriate on vee diagrams; *redundant* entries indicate students' tendency to learn information verbatim and/or in isolation instead of seeking a better re-organization of concept hierarchy or meaningful integrative crosslinks with the first occurrence of concept; whilst *linking word-type entries* are more suitable as linking words, indicate students' difficulties to distinguish between "mathematical concepts" and descriptive phrases. The data in Table 1 and partial views in Figures 1 to 4 show that there was an overall 2.4% increase in valid nodes compared to an associated 7.3% decrease in inappropriate entries. Specific significant increases were noted with the type *concept symbols/expressions* (225%), *mathematical statements/expressions* (63%), *general formulas/expressions* (800%) and *names of relevant methods* (200%) as Nat tried to enhance the meaning of various vertical segments of the concept hierarchy. Figure 1 shows the segments of Nat's first map that attracted critical comments during the first group presentation. His peers felt that it was not illustrating sufficient information to guide a solution of a D.E. problem. Some comments referred to the inappropriate use of important concepts such as *differential form* and *derivative form* as linking words in the propositions: "First Order Linear Differential Equation" *differential form* "M(x,y)dx +N(x,y)dy = 0" and "First Order Linear Differential Equation" *derivative form* "a_0 dy/dx + a_1 y = Q(x)." Over the semester, in response to critiques and feedback from his peers and researcher, these segments evolved to its final form in Figure 2 which shows a more integrated and differentiated network structure that included 9 more new nodes, much higher average hierarchical levels per sub-branch and approximately twice as many multiple branching nodes.

In terms of the structural criteria, data and partial views in Figures 2 and 4 show that Nat's final map had changed significantly compared to the first one in terms of its structural complexity in spite of the unchanged number of main branches, There was a substantial increase (366%) in integrative crosslinks, predominantly more progressive differentiation at various levels (56% increase in multiple branching nodes), 31% increase in sub- branches, 57% increase in average hierarchical levels per sub-branch, and a significant increase (63.6%) in valid propositions. Some of the changes (compare Figures 3 and 4) showed deletions of nodes (*independent variable*, *dependent variable*), merging of two branches (*second order linear differential equation*, *nth order linear differential equation*) to give a more enriched, differentiated and integrated concept hierarchy as partially shown in Figure 4. Nat's propensity to use linking words that are trivial or procedural detracts, rather than add value to the propositions as succinct statements describing the nature of interconnections. These propositions could be further enhanced mathematically with appropriate revision to the linking words.

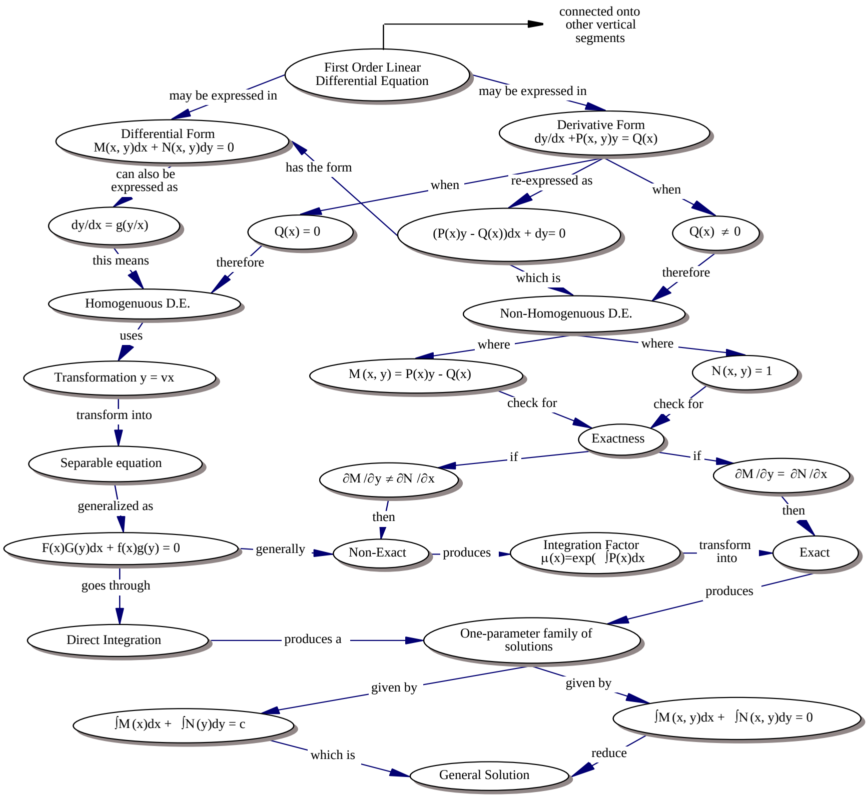

Figure 2. Left vertical segments - Nat's final concept map

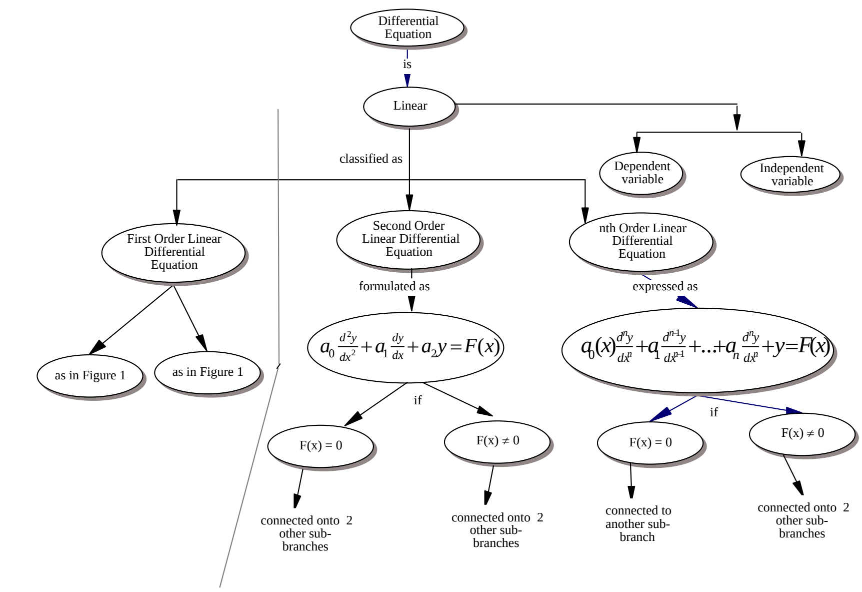

Figure 3. Middle & right vertical segments - Nat's first concept map

3.2 Vee Diagram Data Analysis

The structure of the vee (see Figure 5 for one of Nat's vee diagram) and guiding questions (see Table 2 below) provide a systematic guide to students as they reason from the problem context (EVENT/OBJECT) and given information (RECORDS) to identifying relevant principles, theorems, formal definitions and major rules (PRINCIPLES) and (CONCEPTS) which can guide the development of appropriate methods and procedures (TRANSFORMATIONS) to find an answer (KNOWLEDGE CLAIM) to the (FOCUS QUESTION). As Gowin (1981) points out: "*The structure of knowledge may be characterized (in any field …) by its telling questions, key concepts and conceptual systems; by its reliable methods and techniques of work…*" (pp. 87-88). Therefore, a vee diagram can be a useful tool for not only assessing students' proficiency in solving a problem but it can assess the depth and extent of its theoretical basis by requesting students to identify mathematical principles and concepts underpinning the methods. Thus vee diagrams are assessed qualitatively in terms of one overall criteria (the appropriateness of entries in each section according to the guiding questions in Table 2 and given problem) and a more specific one (the extent of integration and degree of correspondence between listed principles and listed main steps).

3..1.1 SECTIONS	**GUIDING QUESTIONS**
Theory	What theory(ies), major principles govern the methods?
Principles	How are the concepts related? What general rule, principle, formula do we need to use?
Concepts	What are the concepts used in the problem statement? Relevant concepts required solve problem?
Event/Object	What is the problem statement?
Records	What are the "givens" (information) in the problem?
Transformations	How can we make use of the theories/principles/concepts/records to determine a method?
Knowledge Claim	What is the answer to the focus question given the event?
Focus Question	What is the problem asking about?

Table 2: Guiding Questions for Vee Diagrams of Mathematics Problems

Nat's vee diagrams were drawn after one cycle of group and 1:1 critiques, which means that he was able to use his first revised concept map to guide his vee diagram constructions. The first problem (Figure 5) was on first order whilst the other 3 were all on second order D.E. of type non-homogenous with constant coefficients (P2), homogenous with constant coefficients (P3) and homogenous with variable coefficients (P4). In all four vee diagrams, the common entry under THEORY was "differential equations" with a second one reflecting the general order (n = 1 or 2) of the D.E. A third entry of "homogenous with constant coefficients" was included for P3, and for P4, additional entries were "homogenous" and "power series." For the sections, EVENT/OBJECT, FOCUS QUESTIONS, and RECORDS all entries were appropriate and extracted directly from the problem statements and according to the guiding questions of Table 2. In contrast, Nat's selections of entries under CONCEPTS included others not explicitly in the problem statement but were considered relevant. This means that, Nat already recognizes potential underlying principles, and guided by his revised concept map, he selected the most "suitable" principles, and subsequently relevant concepts, for each vee diagram. For example, in Figure 5, his concept list was: {exactness, derivative, integrating factor, partial differentiation, *initial value* problem, *general solution*, *particular solution*} where 3 of the 7 concepts listed were explicitly stated in the problem statement whilst the rest were inferred as being relevant; similarly, for the other 3 vee diagrams. However, entries for PRINCIPLES required some reflection and consideration. For the first problem, his initial diagram only included 3 of the listed principles. A fourth one was added in the final version when he realized that none of the 3 listed principles justified part v under TRANSFORMATIONS. This was a positive self-realization indicative of a growing confidence in his skills to complete the vee diagram. However, he did not pick up the missing principle underlying the normalization step in part i of the transformations. The rest of the main steps had corresponding principles supporting their transformations. As a presenter of his own work and critic of peers' maps, he was aware that maintaining a close correspondence between principles and main steps of the solution was a critical aspect of, and a common problematic area when constructing a vee diagram.

The specific criteria of a tighter integration and correspondence between main steps and principles calls into consideration the inclusion of relevant principles and the "statement," or "wording" of identified principles. That is, are the listed principles stated in theoretical terms (i.e. formal definitions or general rules) and not as procedural instructions? An inspection of Nat's vee diagrams shows a number of listed principles have the ingredient concepts; however the statements are phrased in procedural or algorithmic terms. For example, in the vee diagram for P3, Nat listed principles (iii) and (iv) as: "*with successful substitution forms:* $a_0m^2 + a_1m + a_2 = 0$ *which is called 'auxiliary equation'*" and "*roots* m_1 *and* m_2 *are obtained after solving the auxiliary equation* *for m*" respectively. These statements are more procedurally worded (*substitution* and *obtained after solving*) than conceptual. This algorithmic nature is also evident in the choice of linking words (*substituted to give*, *must satisfy*, and *check for*) used in the partial segment shown in Figure 4. It appears that his background in computer programming and experience with flow charts are influencing his perceptions of what constitutes appropriate "linking words" and "principles." Alternatively, this procedural view of mathematics suggests that his perceptions of "mathematical knowledge" up to this point may have been predominantly as "a collection of methods and procedures." Thus, in spite of the improvement in aligning the listed principles with main steps of solution in vee diagrams, Nat still needs to rephrase his "procedural" principles to be more theoretical.

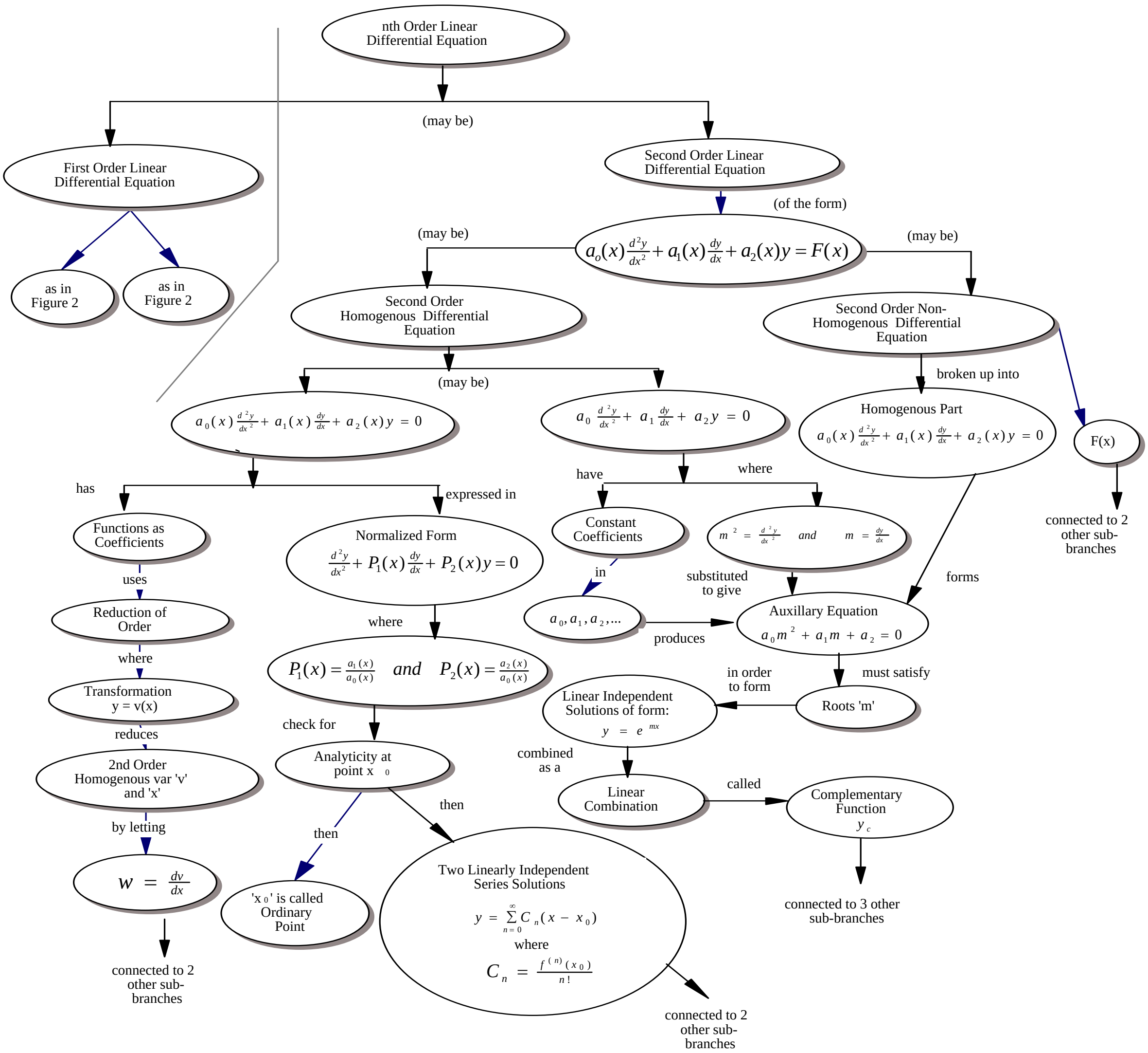

Figure 4. Middle & right vertical segments. Nat's final concept map

4 Summary

Both Nat's final concept maps and final vee diagrams showed significant changes by the end of study compared to his initial attempts. For example, the final map was structurally more complex with increased integrative crosslinks, more progressive differentiation links, additional concepts, and increased number of valid propositions. Evidently, Nat's understanding of differential equations had become more structured, better organized and more enriched to the point that it greatly facilitated the construction of his vee diagrams. For example, he wrote: "*the vee diagram consists of the theoretical and practical senses of the event/object … directed by the concept map (right segment, Figure 4) flow … these senses help guide the transformations.*" That is, he was beginning to realize for himself the value of a clearly organized concept hierarchy as a guide to make decisions about effectively solving a problem. Since his vee diagrams were constructed after the first cycle of critiques, they attracted relatively less criticism during critiques unlike his concept maps. From the sociocultural perspective, Nat's conceptual, more structured and better organized understanding of D.E. was partially a result of sustained social interactions, and critiques from his peers and researcher throughout the semester. For example, Nat wrote: "*Due to questions raised in class, on what 'auxiliary equations' represented and how it's formed. Therefore, more details were presented …*" and "*the whole concept of reduction of order needed clarification in all senses due to its shortened presentation before.*" With reference to two of his vee diagrams, he noted: "*I encountered very little or no critical comment during class presentations ... (so)… I'm left with no urge to make any further modifications … not criticized at all, therefore no changes.*" That is, his determination to minimize critical comments from others motivated Nat to continually develop a more complete and comprehensive map. For example, he added branches to clarify the concepts: *auxiliary equation, reduction of order* in his second map, and added a new branch in final map to accommodate *power series solutions* (see Figure 4).

METHODOLOGICAL SIDE

4 THEORIES
Differential Equations
First Order Linear Differential

3 FOCUS QUESTION
What are the general and particular solutions of the given differential equation?

2 KNOWLEDGE CLAIM
The general solution of the D.E. is: $x^2 y = -\frac{1}{x} + c$ and particular solution is: $x^2 y = -\frac{1}{x} + 1$

5 PRINCIPLES

1. The derivative form of 1st order linear D.E. is: $\frac{dy}{dx} + P(x)y = Q(x)$……. (1) where P(x) and Q(x) are functions of x.
2. Expressing (1) in the form:
   M(x, y)dx + N(x, y)dy = 0
   $\Rightarrow$[P(x)y – Q(x)]dx + 1 dy = 0 where
   M(x,y) = P(x,y)y – Q(x) & N(x, y) = 1.
   (1) is an exact Differential Equation if:
   $\frac{\partial y}{\partial x} = \frac{\partial N}{\partial x}$ otherwise "NOT EXACT."
3. If not exact, an integrating factor $\mu(x)$, is used to transform it into an "EXACT" D.E.:
   $$\mu(x) = e^{\int P(x)d(x)}$$
4. The given initial conditions are applied to the general solution to obtain the particular solution.

6 TRANSFORMATIONS

(i) Normalize $x^4 \frac{dy}{dx} + 2x^3 y = 1$... (2) to obtain the form:
$\frac{dy}{dx} + \frac{2}{x} y = \frac{1}{x^4}$ by dividing (2) by x^4.

(ii) $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ Not Exact.

(iii) Seek an integrating factor:
$\mu(x) = e^{\int P(x)d(x)}$
$\therefore$ $\mu(x) = x^2$

(iv) Multiply (2) by $\mu(x) = x^2$
$\Rightarrow$ $x^2 y = -\frac{1}{x} + c$ …. (3)

(v) Apply initial conditions given: f(1) = 0, $\Rightarrow$ $(1)^2(0) = -\frac{1}{1} + c$
c = 1 $\Rightarrow$ $x^2 y = -\frac{1}{x} + 1$

8 CONCEPTS
Exactness, derivative form, integrating factor, partial differentiation, initial value problems

7 RECORDS
$x^4 \frac{dy}{dx} + 2x^3 y = 1$, $y(1) = 0$

9 EVENT/OBJECT
Solve the following differential equation and obtain the general solution, and use the given initial conditions to obtain the particular solution: $x^4 \frac{dy}{dx} + 2x^3 y = 1$, $y(1) = 0$.

Figure 5. Nat's vee diagram - differential equation problem 1

From the perspectives of Richards (1991) and Knuth & Peressini (2001) Nat and his peers engaged in inquiry mathematics where classroom discourse was both "to convey meaning" and "to generate meaning." Nat, through the use of maps/diagrams engaged in mathematical discussions, dialogues and critiques with his peers and researcher. His fluency with the language of D.E. and the communication of this mathematical understanding publicly was made more effective and efficient with maps/diagrams. There was noticeable growth in Nat's in-depth understanding of D.E. as indicated by the increased number of valid propositions and structural complexity of conceptual networks in his final map, and improved correspondence between listed principles and solutions on vee diagrams. However, because of his tendency to view interconnections and principles with a procedural bias, Nat needs to continuously revise his maps/diagrams over a longer period of time with appropriate critiques and feedback from other mathematics people. Finally, the established socio-mathematical norms, and use of maps/diagrams appeared to promote a classroom environment that was alive with meaningful discussions as students engage in the critiquing and justification processes. Whilst it could be argued that this would happen irrespective of the type of metacognitive tools used, the author proposes that the unique visual structures of the maps/diagrams were pivotal in facilitating and promoting dialogues and critiques. Evidently, the established socio-mathematical norms and use of the metacognitive tools had substantially influenced Nat's perceptions of mathematics as reflected by the progressive improvement in his maps/diagrams. Clearly, with the right supportive classroom environment, students can be encouraged to dialogue, discuss and communicate mathematically. In doing so, students begin to realize that learning mathematics meaningfully involves much more than implementing a sequence of steps. Findings from this case study suggest that students are amenable to changes in classroom practices, and with their cooperation, and using appropriate metacognitive tools, mathematics learning can be made more enriching and meaningful at an individual level.

5 References

Afamasaga-Fuata'i, K. (2003a). *Numeracy in Samoa: From Trends & Concerns to Strategies.* Paper presented at the Samoa Principals Conference, Department of Education, EFKS Hall, Samoa, January 28-30, 2003.

Afamasaga-Fuata'i, K. (2003b). *Mathematics Education: Is it heading forward or backward?* Paper presented at the Measina Conference, Institute of Samoan Studies, National University of Samoa. December 4, 2003.

Afamasaga-Fuata'i, K. (2002a). *Vee diagrams & concept maps in mathematics problem solving.* Paper presented at the Pacific Education Conference (PEC 2002), Department of Education, American Samoa, July 23, 2002.

Afamasaga-Fuata'i, K. (2002b). *A Samoan perspective on Pacific mathematics education.* Keynote Address. Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia (MERGA-25), July 7-10, 2002 (pp. 1-13), University of Auckland, New Zealand.

Afamasaga-Fuata'i, K. (2001). *Enhancing students' understanding of mathematics using concept maps & vee diagrams.* Paper presented at the International Conference on Mathematics Education (ICME), Northeast Normal University of China, Changchun, China, August 16-22, 2001.

Afamasaga-Fuata'i, K. (2000). *Use of concept maps & vee maps in mathematics problem solving.* Paper presented at the School of Education, Faculty of Education & Community Service Seminar Series, University of Reading, UK, June 8, 2000.

Afamasaga-Fuata'i, K. (1999). Teaching mathematics and science using the strategies of concept mapping and vee mapping. *Problems, Research, & Issues in Science, Mathematics, Computing & Statistics*, 2(1), 1-53. Journal of the Science Faculty at the National University of Samoa.

Afamasaga-Fuata'i, K. (1998). Learning to Solve Mathematics Problems Through Concept Mapping & Vee Mapping. National University of Samoa.

Ausubel, D. P., Novak, J. D., & Hanesian, H. (1978). *Educational Psychology: A Cognitive View*. New York: Holt, Rhinehart and Winston. Reprinted 1986, New York: Werbel & Peck.

Ernest, P. (1999). Forms of knowledge in mathematics and mathematics education: philosophical and rhetorical perspectives. *Educational Studies in Mathematics*, 38, 67-83.

Gowin, D. B. (1981). *Educating*. Ithaca, NY: Cornell University Press.

Knuth, E., & Peressini, D. (2001). A theoretical framework for examining discourse in mathematics classrooms. *Focus on Learning Problems in Mathematics*, 23(2 & 3), 5-22.

Liyanage, S. & Thomas, M. (2002). *Characterising secondary school mathematics lessons using teachers' pedagogical concept maps.* Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia (MERGA-25), July 7-10, 2002 (pp. 425-432), University of Auckland, New Zealand.

Mintzes, J. J., Wandersee, J. H., & Novak, J. D. (Eds.) (1998). *Teaching Science for Understanding. A Human Constructivistic View.* San Diego, California, London: Academic Press.

Novak, J. (2002). Meaningful learning: the essential factor for conceptual change in limited or appropriate propositional hierarchies (LIPHs) leading to empowerment of learners. *Science Education*, 86(4), 548-571.

Novak, J. D. (1998). Learning, Creating, and Using knowledge: Concept Maps as Facilitative Tools in Schools and Corporations. Mahwah, NJ: Academic Press.

Novak, J. D. (1985). Metalearning and metaknowledge strategies to help students learn how to learn. In L. H. West, & A. L. Pines (eds.) *Cognitive Structure and Conceptual Change* (pp. 189-209). Orlando, FL: Academic Press.

Novak, J. D. & Gowin, D. B. (1984). *Learning How to Learn*. Cambridge University Press.

Richards, J. (1991). Mathematical discussions. In E. von Glaserfeld, (ed.), *Radical constructivism in mathematics education* (pp. 13-51). London: Kluwer Academic Publishers.

Ruiz-Primo, M. A. & Shavelson, R. J. (1996). Problems and issues in concept maps in science assessment. *Journal of Research in Science Teaching*, 33(6), 569-600.

Steffe, L. P., & D'Ambrosio, B. S. (1996). Using teaching experiments to enhance understanding of students' mathematics. In D. F. Treagust, R. Duit, & B. F. Fraser (Eds.), *Improving teaching and learning in science and mathematics* (pp. 65-76). Teachers College Press, Columbia University, New York.

Williams, C. G. (1998). Using concept maps to access conceptual knowledge of function. *Journal for Research in Mathematics Education*, 29(4), 414-421.