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USES OF CONCEPT MAPPING IN TEACHER EDUCATION IN MATHEMATICS

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Abstract A case study of the concept maps of two pre-service teachers illustrates the potential of concept mapping to the teacher educator. The maps reveal much about whether future teachers grasp the nature of mathematics as a conceptual system, understand the conceptual content of mathematical procedures, and possess the requisite pedagogical content knowledge to mediate such understandings to future learners. The map of one of the two teachers reveals that she possesses these understandings. The map of the other shows a formalistic understanding of mathematics. Concept mapping also functions as an epistemological heuristic for pre- and in-service teachers.

1 Case Study

Despite the emphasis on conceptual understanding that characterizes the reform standards of the National Council of Teachers of Mathematics, concept mapping in mathematics has been underutilized in the US. This is unfortunate, since it has the potential to begin to counteract the superficial treatment of concepts occasioned by the failure to develop a coherent curriculum that identifies essential concepts and probes them in sufficient depth. In addition, reform mathematics curricula have all but abandoned the teaching of algorithms, preferring to consign computation to calculators instead (Morrow & Kenney, 1998). This practice has not only been widely criticized (Hu, 1999; Schmittau, in press), but, has in fact, played a major role in fueling the US "math wars". At the root of this false dichotomy between mathematical procedures and mathematics concepts is the notion that algorithms can only be taught mechanically, as they often were in the past, and are therefore, incompatible with the teaching of concepts. Algorithms, however, are fully conceptual cultural historical products and should be taught as such. Concept mapping can serve as a useful tool to enable the linking of algorithms with their conceptual content.

The following case study of the concept maps of two pre-service teachers illustrates this point, and provides an example of the manner in which a concept map can alert the teacher educator to whether or not students are understanding mathematics as a conceptual system (in which procedures are fully integrated), or grasping it at the level of mere formalism (Schmittau, 2003). Both students drew maps of their understanding of the concept of multiplication, subsequent to extensive class discussion of the concept. Since the maps were quite large, space considerations limit presentation to small sections of each.

In Figure 1 A, a section of Katie's map reveals her understanding of what multiplication *is,* viz., "a change in units in order to take an indirect measure" (cf. Davydov, 1992). Shawn, however, sees multiplication as an "operation which is composed of" an "operator" and "operands" (Figure 1B). Katie's definition can be meaningfully taught to children in the early elementary school years (Davydov, 1992), while Shawn's would make very little sense to students prior to college mathematics.

Katie's definition reflects the development of multiplication from the need to take a measurement or count of a quantity of objects or units sufficiently numerous to render a direct count tedious and subject to error. In such a circumstance, it is advantageous to construct a larger unit (or multiple) of the unit of interest, and use it to take an indirect count. This is done in the case of the area of a rectangle, for example, when the number of unit squares in a row are counted, and then the number of rows are counted in order to indirectly obtain the area as the number of square units. Such an understanding of multiplication underlies the algorithm for obtaining the product of multi-digit numbers. It does, however, require that the products of single digit numbers (multiplication "facts") be committed to memory. It is not sufficient that the calculator can call up these "facts"; they must be stored in the human memory if they are to be recognized in subsequent mathematical studies in the myriad of conceptual interrelationships into which they enter. Katie refers to these in her concept map, indicating that their commitment to memory is important for mathematical understanding, and should not be by-passed as is often now occurring in the wake of the reform movement. All of these points were emphasized during our extensive class discussions of the concept of multiplication and the various numerical domains in which it is defined. During these we noted the inadequacy of the "repeated addition" definition that is ubiquitous in US textbooks (Schmittau, 2003). Accordingly, neither Katie nor Shawn invoked this notion in their concept maps of multiplication.

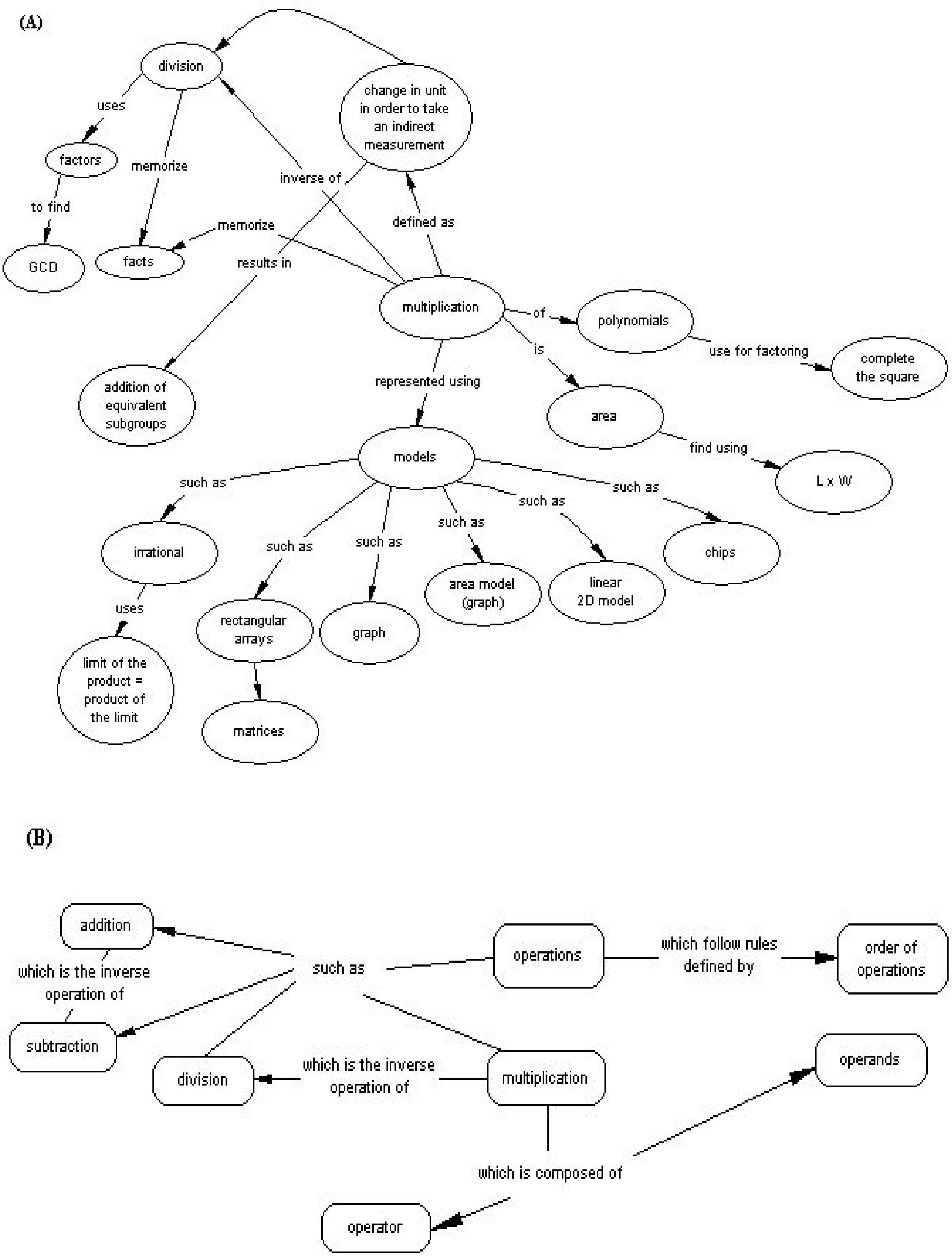


Figure 1. Katie's (A) and Shawn's (B) maps of the concept of multiplication.

In Figure 2, a portion of Katie's map reflects the centrality of the concept of area to her understanding of multiplication. In the case of rectangular area, the unit changes from a unit square to a row of such squares, which can then be counted to obtain the area in square units. Katie also presents area models for the conversion of a sum of terms (polynomial) to a product (factoring) following the methods of Al-Khowarizmi (Karpinski, 1915). Such area models are the conceptual foundation for the completion of the square. The upper model for the solution of the quadratic equation "$x^2 + 10x = 39$" shows Al-Khowarizmi's method, while the lower model displays the solution using algebra tiles. In his development of algebra 1000 years ago, Al-Khowarizmi solved this equation by first drawing a square (having dimensions x by x), then dividing 10 by 4, and using the result to add a rectangle of dimensions "x" by 2.5 to each side of the square. There were as yet no algebraic symbols, but since it is cumbersome to use word labels for variables, in my presentation to Katie and Shawn's class, I labeled the length of these four rectangles "x". Al-Khowarizmi then literally completed the square (the new larger square, that is) by adding the four small squares in each corner. Each of these had an area of 2.5 x 2.5 = 6.25, and since there were four of them, their total area was 25 square units. Now the original equation (again we are using symbols that Al-Khowarizmi lacked) becomes $x^2 + 10x + 25 = 39 + 25 = 64$. Since the length of a side of the square is $x + 5$ and the area of the square is 64, the square has dimensions 8 by 8, and $x + 5 = 8$. Hence, x is 3. (Al-Khowarizmi's geometric method did not permit negative solutions.)

Katie's lower model shows an algebra tile model for the same problem. Algebra tiles are a modern manipulative designed to geometrically model polynomials. Although the ten "x by 1" rods cannot be broken (to obtain rods having a dimension of 2.5 as in Al-Khowarizmi's solution), they can be arranged to produce a new larger square by placing five of them along the side and five along the bottom of the x by x square and completing the new square with 25 unit squares.

Katie's inclusion of these models in her map is important for several reasons. First, the map reveals area as the central antecedent concept necessary for an understanding of factoring. Many secondary students fail to grasp factoring despite the use of algebra tiles, and teachers typically do not understand why a model that is so transparent to them is not equally so for their students. If teachers realize that area is insufficiently understood by many students (Schmittau, 2003), they will understand why area models sometimes fall short of their anticipated effect. Second, Katie's map reveals a knowledge on her part of the cultural historical development of the solution of the quadratic equation by factoring, as well as its current rendering by the use of a popular manipulative. Her map reveals that she has internalized from the class presentations, the relevant content and pedagogical content knowledge necessary to teach this concept meaningfully.

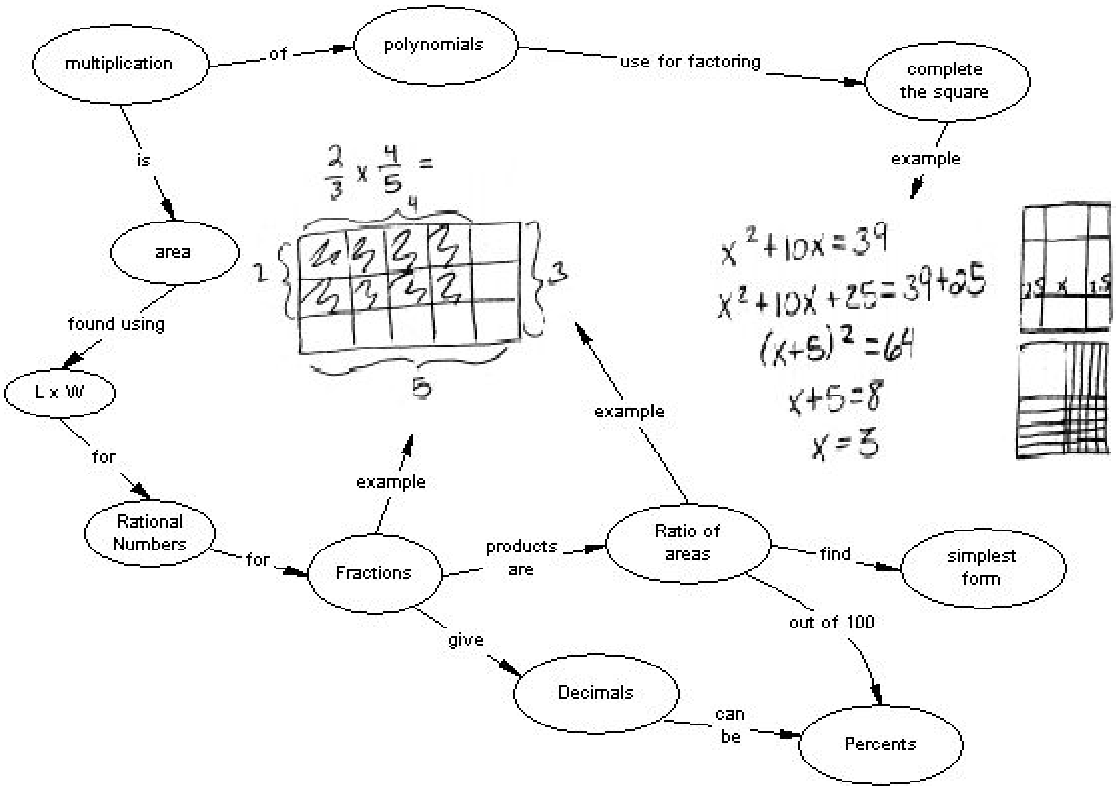


Figure 2. Katie's map showing the historic role of the concept of area in the development of algebra and the factoring of polynomials, and the concept of area underlying the algorithm for the multiplication of fractions.

Some teachers resist the use of algebra tiles, believing that a solution method that is purely "algebraic" (i.e., symbolic), is just as effective. However, the fact that Al-Khowarizmi tested his invented algebraic methods against geometric models and that geometric models were used up to the nineteenth century, should caution against the tendency to omit this important step in concept development. Further, Al-Khowarizmi invented this method one thousand years ago, some 500 years before the creation of algebraic symbols. So immediately engaging students at the level of symbolic expression omits from their ontogenetic experience the equivalent of hundreds of years in the phylogenetic development of this concept. Such an approach can scarcely be considered a recipe for adequate conceptual understanding.

If we explore this portion of Katie's map further (Figure 2), we see an area model for the multiplication of fractions also, in which the algorithm for the product of two fractions is identified by Katie as a "ratio of areas". Katie now understands what virtually none of my graduate students in mathematics understand prior to our class discussions, viz., that the reason why the algorithm for the product of two fractions "works", is because the product of the numerators and the product of the denominators are *areas* whose ratio is the product of the fractions. In the example used by Katie, viz., "2/3 x 4/5", the shaded area of the rectangle represents the product of the numerators "2 x 4", while the area of the larger rectangle is the product of the denominators "3 x 5". Their ratio, 8/15, is four-fifths of two-thirds, the product of the two fractions.

Shawn (Figure 3) states simply that "integer fractions" … "have the multiplication formula a/b x c/d = ab/cd which is a ratio of area". Without further elaboration or a model representation, it is difficult to discern from his map alone whether he grasps the precise nature of this ratio. This propositional acknowledgement alone, however, is more than is commonly perceived by the typical pre-service or in-service teacher.

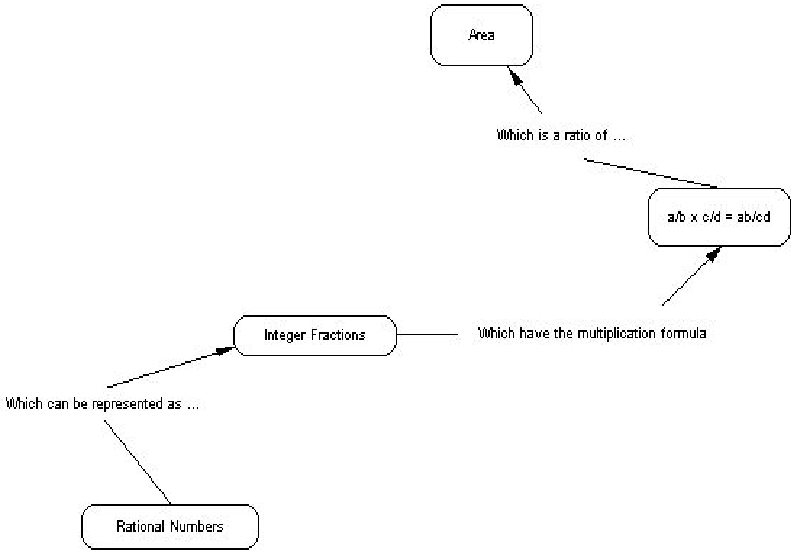


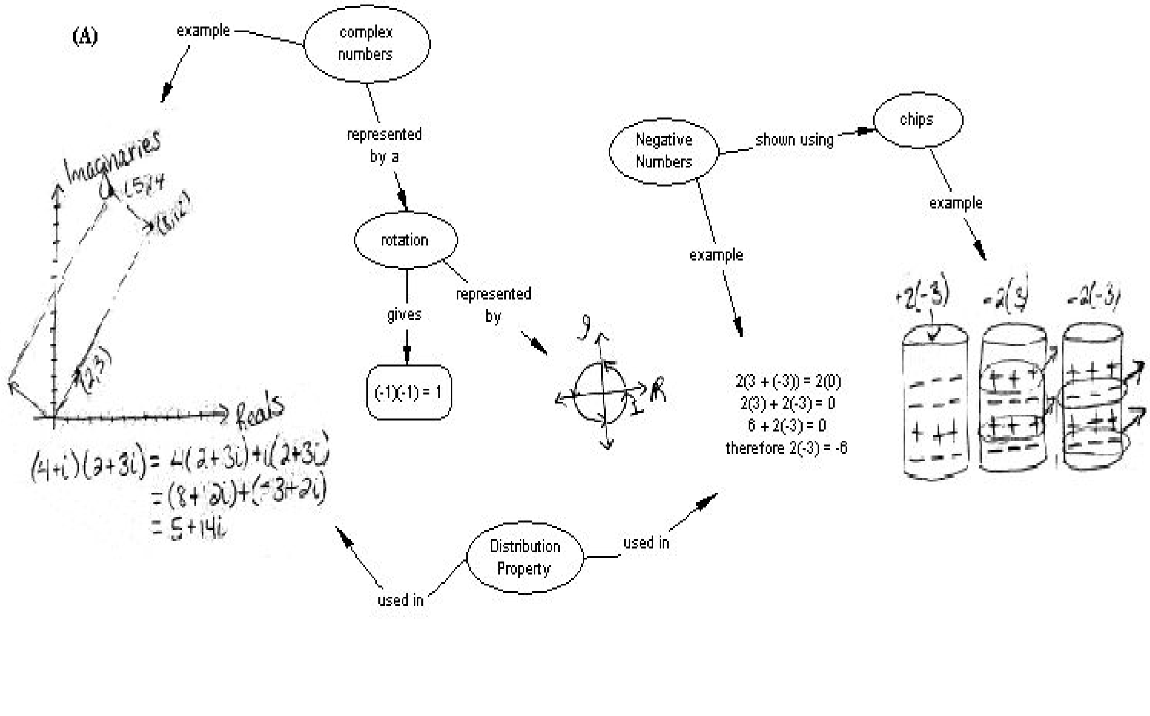
Figure 3. Shawn's map showing a formalistic understanding of fraction multiplication.

Finally, the current reform movement in mathematics often advocates that teaching go no further than the use of manipulatives, and then allow students to "construct" their own algorithms (Morrow & Kenney, 1998). There is, however, no certainty that such student constructed algorithms will be correct. This does not seem to matter to the reformers, who appear unable to connect the powerful culturally and historically constructed algorithms with their conceptual content. Their answer is to abandon the teaching of such general methods in favor of dealing only with "concepts", a practice that relegates procedural knowledge further to the rote end of the meaningful-rote learning continuum (Novak & Gowan, 1984), reducing it to little more than a memorized sequence of calculator keys. Katie's map, however, reveals that she is aware of the development of this concept in the historical progression of mathematical knowledge, and furthermore, possesses the pedagogical content knowledge necessary to teach it effectively to students.

Another aspect of multiplication students find incomprehensible concerns the product of two negative numbers. In Figure 4A, Katie's map indicates that multiplication of negative numbers can be modeled using chips to represent positive and negative charges. Such a model makes use of Ausubel's concept of an advance organizer, which is often necessary because of the early grades (frequently 5^{th} or 6^{th}) at which this topic is introduced. Here, the notion of charged particles serves this function. In the model showing containers of charged particles, Katie first shows +2(-3). Beginning with 3 positive and 3 negative charges in the container on the left (for a net charge of zero), two groups consisting of 3 negative charges in each, are added, resulting in a net charge of –6. From the middle container (starting with 6 positive and 6 negative charges), two groups of 3 positive charges are removed, representing –2(+3) and leaving a net charge of -6. From the container on the right (again starting with 6 positive and 6 negative charges), two groups of 3 negative charges are removed, leaving a net charge of +6. Such a use of an advance organizer is imperative, I argue in our class discussions, because the conceptual integration of the product of two negatives is to be found in the system of complex numbers, which is not studied until high school, long after this topic has been taught.

Accordingly, Katie's map shows a real understanding of the conceptual connections I presented to her class, linking the product of two negatives to the complex number system. Her map displays a linkage from negative numbers to the product "(-1)(-1) = 1" which is linked to multiplication of "complex numbers" and "represented by a rotation" shown in the complex plane. This is the actual inception of the concept of multiplication by a negative number, and Katie's representation of (4 + i)(2 + 3i) in the complex plane together with her assertion that the "distributive property" is "used in" obtaining this product, suggests an understanding that multiplication by the scalar quantity "4" repeats the vector (2,3) four times, quadrupling its norm (length) and resulting in the vector (8,12). Multiplication of (2,3) by "i", however, in a decided break with the meaning of multiplication for real numbers, produces a 90^o counterclockwise rotation of this vector, resulting in the vector (-3,2). In class I showed students that the vector (1,0) when multiplied by "i" rotates to the vector (0,1). Multiplying by "i" again results in the vector (-1,0). Hence, the fact that (-1)(-1) = 1 is due to the fact that –1 multiplied by itself reflects two further 90° counterclockwise rotations, i.e., the vector (-1,0) is rotated 180 degrees to (1,0). Katie's model shows four 90 degree counterclockwise rotations to produce the real number "1", which she states is "(-1)(-1)". This, together with linkages to the graph of the product of two complex numbers obtained using the "distributive" property, connected to "complex number" that are "represented by a rotation" modeled in the complex plane, suggests that she has internalized what was taught in my graduate course, and has both the relevant content knowledge and pedagogical content knowledge to teach this concept with meaning. A teacher who possesses these understandings may be expected to point out the connection to multiplication of negative numbers when multiplication of complex numbers is taught. None of my graduate students in mathematics have ever made this connection. They learn it for the first time in my graduate course.

Shawn's map deals with this topic very minimally, without models. His map states that for complex numbers "multiplication is defined by… (a+bi)(c+di) = (ac – bd) + (cb + ad)i". This section of his map is unconnected to the section dealing with "negative numbers" (Figure 4B). Here he states that "When a and b are both negative… a x b = (-a) – (-a) - …", that is, "-a subtracted from itself b times". If that is the case, then (-3)(-2) = (-3) – (-3) = 0, rather than +6. Hence, despite the fact that several meaningful ways of modeling this concept were presented during our class discussions, Shawn's map suggests that he views multiplication of two negatives as repeated subtraction, which is an inaccurate conceptualization.



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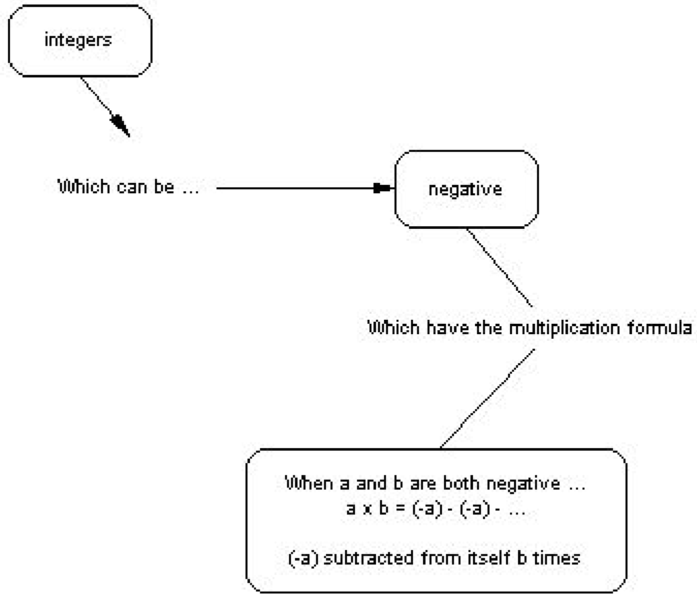

Figure 4. Katie's (A) and Shawn's (B) differing understandings of the multiplication of two negatives.

Shawn and Katie were both present in the same classes in which I taught the conceptual content of this concept that is reflected in Katie's map above. However, the evidence from their maps is that one internalized the concept in its systemic interconnections, while the other continued to see it through a formalistic lens. Katie's map gives evidence that she possesses the requisite conceptual understanding, and historical and pedagogical content knowledge, to mediate the concept of multiplication meaningfully to students, without separating its so-called "procedural" from its "conceptual" content. Indeed, it appears that she can move rather seamlessly between the two. Shawn's map, in its entirety, has considerable extension, encompassing multiplication of matrices, determinants, and the cross products of vectors. But the connections are consistently formalist and give no evidence that his teaching will go beyond a formalistic approach. Both students are nearing completion of the masters' degree, but their maps reveal very different understandings of this fundamental concept.

2 Epistemological Value

While Ausubelian theory emphasizes the conceptual connections that are requisite for meaningful learning (Novak & Gowin, 1984), Vygotskian theory points to the need to unfold the historically developed conceptual content from its encapsulation in symbolic expression in order to pedagogically mediate the full restructuring of the concept (Davydov, 1990). In the examples above, the concept maps produced were made subsequent to class discussions and presentations in which I shared my own conceptual and historical analyses of multiplication with masters' level students preparing to be high school teachers. I typically require that doctoral students conduct such analyses on their own and frame pedagogical recommendations for the improvement of instruction based upon their findings. James Vagliardo's analysis of the concept of logarithm is illustrative of the role of concept mapping in this process (cf. his conference contribution).

In addition, in my work as lead mathematics educator in the Teacher Leadership Quality Partnership in New York state, I use concept mapping to reveal to in-service teachers why it is imperative that they teach topics not contained in their textbooks or the state mathematics curriculum. In mapping mathematical concepts taught in middle school and searching for their conceptual roots, it becomes clear that the concept of bases, for example, is a central antecedent concept with the power to render more effectively the meaning of such concepts as decimals, fractions, and polynomials. But the concept of bases cannot be attained by studying base ten (Vygotsky, 1986); for adequate conceptualization of such a superordinate concept, the study of multiple bases is required. The folly of superficially covering many topics is simultaneously revealed; only by establishing a conceptual base of concepts central to the future development of mathematics, can students begin to grasp the nature of mathematics as a conceptual system. Yet fifteen years into the reform movement, US curricula continue to cover too many topics each year, to repeat the same topics year after year, and with little increase in depth (Schmidt, Houang, & Cogan, 2002). It would seem that the reform movement in mathematics could benefit from the pedagogical potential of concept mapping.

3 Acknowledgements

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