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SUBSTANTIVE KNOWLEDGE AND MINDFUL USE OF LOGARITHMS: A CONCEPTUAL ANALYSIS FOR MATHEMATICS EDUCATORS

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Abstract. This study uses concept mapping to investigate the logarithm. Empirical evidence documents the extent to which selected instructors and students command substantive knowledge about logarithms and analyzes their ability to make mindful use of their current understanding. Initial mapping reflects inadequacy and provides the basis for an in-depth search of the cultural and historical context that gave rise to the logarithm concept. The author presents a map that incorporates the essence of the concept from a cultural historical perspective as its central structure. This key ideational relationship, the conceptual cross link between arithmetic and geometric sequences discovered by John Napier, precisely identifies the source of weakness in conceptualization empirically evidenced among faculty and students. The improved map visually provides the understanding necessary for corrective intervention and is the prime reference in the development of a pedagogical approach toward more substantive knowledge and mindful use of the logarithm concept. Study results indicate that concept mapping can provide an epistemological tool for sound curricular and instructional development in mathematics education; one that seeks to locate and build on the essence of conceptual foundations. In a scientific discipline, graphically rendering such substantive cognitive structures maximizes the probability of their mindful use in related mathematical reasoning.

1 Introduction

Mindful use of an important mathematical concept necessitates substantive knowledge, knowledge that extends well beyond the rote acquisition of standard mathematical procedures. Inversely, mindless use usually involves weak or nonexistent conscious awareness of purpose or meaning involved in activity. Langer (1989) suggests that such mindlessness may be rooted in the development of automatic behavior through repetition and practice. Premature cognitive commitment on the part of a learner, a commitment to an early understanding that lacks the full development that can be achieved through thoughtful contemplation and study of the underlying concepts involved from a historical perspective, may also be the cause. Mindlessness can be induced by organizations with an orientation focused on outcomes, with minor attention given to conceptualization and a focused dependency on rote learning.

Substantive knowledge refers to knowledge that reveals the essence of the concept in question. This notion necessarily avoids the misconception that the cultural historical context which gives rise to an idea, especially a mathematical concept, is of little importance in the development of a deep understanding of that idea. On the contrary, substantive knowledge is grounded in such considerations. An intellectual dedication to a continual search for new and deeper understanding, relating new conceptualizations to current knowledge, may be dependent upon the purity of initial substantive knowledge. Subsequent thought and study is then more likely to locate conceptual cross links, and may, over time, lead to the emergence of mental models that reflect the essence of the original idea.

2 Concept Mapping and Historical Research as a Combined Epistemological Tool

Concept mapping is at the heart of this study. Investigations of four types: an historical search, a conceptual analysis, clinical research involving mathematics teachers and their students, and the development of a curricular approach to logarithms that addresses the historical and cultural foundation of this mathematical concept was constructively informed and guided by mapping. The logarithm is a concept whose understanding must be mediated by knowledgeable and skillful instruction to be well understood. Results of the clinical component of the study provide evidence of the need for conceptual intervention through improved curriculum design based on concept mapping and historical research as a combined epistemological tool. This is done in the context of the philosophical and theoretical ideas of Lev Vygotsky (1978) and the notions of concept formation and generalization of V. V. Davydov (1990) applied to the development of theoretical scientific thought.

Historical references to the sixteenth and early seventeenth centuries provide the original view of the logarithm revealed by John Napier. In order to better understand the context that gave rise to his work, an extensive reading of the general history of mathematics was completed, starting in ancient Egypt and tracing the development of computational methods and mathematical thought from Greece to India and its subsequent spread to Europe via Arab merchants in the middle ages. The specifics of Napier's thinking cast in the complementary histories of mathematics and philosophy provide the scientific and philosophical foundations for understanding what a logarithm is, what gave rise to the idea, and gives a sense of the important cultural implications of the discovery**.**

A concept map developed to reveal the discovery of John Napier in its essence provides a surprising consequence with far reaching curricular implications for mathematics educators. The central understanding of the logarithm concept illustrated by this map was compared to those generated from interviews with three faculty members and six of their students and the texts they use as resources. A composite map reflecting all understandings and relationships was also created for purposes of contrasting perceptions. The map further displays traditional instructional content, current curricular focus, and implications for the study of calculus, advanced science, and technology. By means of this composite map, six clinical analysis categories were identified and addressed. Of significance here are the categories of “Conceptual Representation,” “Student Competency and Problem Complexity,” and “Application and Importance.” Findings translated into the development of a significant change in pedagogical approach illustrated in a series of introductory logarithm lessons.

3 Conceptual Analysis From a Cultural Historical Perspective

Scientific ideas are, in the Vygotskian sense, ideas that do not occur spontaneously in the human mind as a result of normal everyday experience but require dedicated theoretical analysis.

> If every object was phenotypically and genotypically equivalent..... then everyday experience would fully suffice to replace scientific analysis. ... real scientific analysis differs radically from subjective, introspective analysis, which by its very nature cannot hope to go beyond pure description. The kind of objective analysis we advocate seeks to lay bare the essence rather than the perceived characteristics of psychological phenomena. (Vygotsky, 1978, p. 63)

Scientific ideas spring from and are mediated by a social, cultural, and historical context. John Napier's definition of a logarithm is just such an idea and provides a generalization worthy of historical exploration using the lens of Ausubel's cognitive learning theory as described by Novak and Gowin (1984). I refer to the notions that cognitive structure is *hierarchically organized*, that this structure is *progressively differentiated*, and that the process of *integrative reconciliation* may yield linkage between concepts providing new propositional meaning. The historical record of Napier's work also reveals strong evidence in support of Vygotsky's thoughts on cultural mediation. Napier's discovery is a perfect example of *integrative reconciliation*. His particular linkage of geometry and arithmetic has impacted the world for nearly four hundred years.

3.1 Historical Foundation

In 1619, John Napier wrote, *Mirifici Logarithmorum Canonis Constructio*, in which he explained his idea of using geometry to improve arithmetical computations. This was the breakthrough that accelerated the discoveries of science and led to the creation of new calculating devices, ultimately leading to modern day electronic calculators and computers.

Seeing there is nothing, right well-beloved students of mathematics, that is so troublesome to mathematical practice, nor doth more molest and hinder calculators, than the multiplication, division, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances. - John Napier

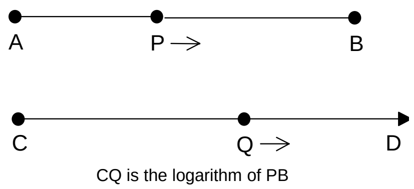

Figure 1. John Napier’s development of the logarithm concept.

The account of the mathematical consideration Napier used in defining a logarithm refers to points moving on two different lines (Figure 1). Consider that point P starts at A and moves along segment AB at a speed proportional to its remaining distance from B. Simultaneously, point Q departs from C and moves along ray CD with a constant speed equal to the starting speed of P. Napier called the distance CQ the logarithm of PB. This idea proved to be an important benchmark in the history of mathematical thought, providing a cross link between concepts that immediately accelerated the interests of science and economics.

The genesis of Napier's discovery is situated in Egyptian, Greek, Hindu, and Arabic thought and reflects the influence of the Renaissance on scientific thinking. From the Egyptian papyrus of Ahmes we see work on the reduction of fractions of the form 2/(2n+1) into a sum of fractions with numerators of one, an early form of computational efficiency. As the flow of ideas is transmitted between cultures, the Greeks provide theoretical structure to mathematical thought. Pythagoras establishes the twice split view of mathematics (Figure 2) (Turnbull, 1969) that places arithmetic and geometry on distinctly separate branches. Hindu mathematics, like the Greek, considered arithmetic and geometry as separate categories of mathematics. It is a significant conceptual separation that becomes the focus in the late sixteenth century; ideas actually cross-linked by Napier.

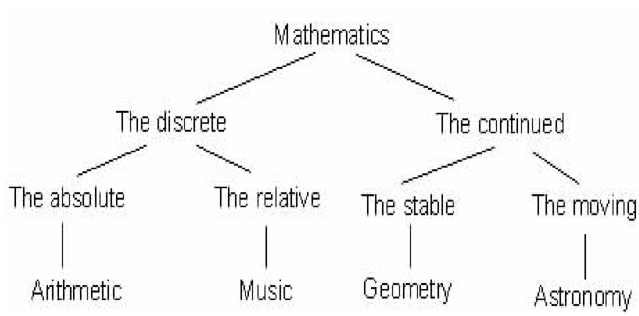

Figure 2. The view of mathematics attributed to Pythagoras.

This historical sampling of mathematical thought reveals the setting, motivation, and approach that made the discovery of logarithms possible. The intent of historical reference is to have a sense of what John Napier knew at the time of his discovery. Access to his understandings sharpens the focus of our own as we consider the true nature of what he revealed to the scientific world. The cultural historical approach to the appropriation of knowledge is dependent on such considerations. Significant is the realization that it is only through this historical lens that the true essence of the scientific thought can be understood. In the case of the logarithm, evidence serves to inform us that, motivated by computational efficiency, Napier provided a theoretical link between the worlds of geometry and arithmetic. This novel generalization provided fertile territory for new development, practical and theoretical.

3.2 Conceptual Essence of the Logarithm Concept from a Cultural-Historical Perspective

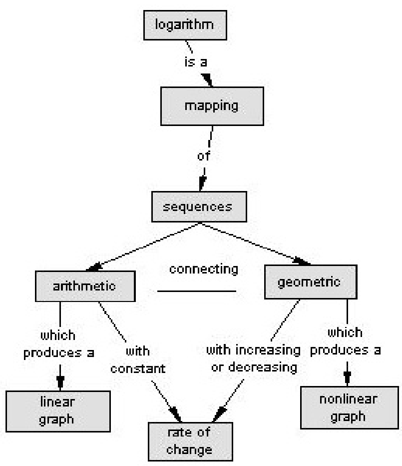

Figure 3. A concept map of logarithm showing the historical conceptual cross link, the genesis of a new mathematical idea.

Based on historical reference, a logarithm is a mapping between number sequences with different types of change rates (Figure 3). The arithmetic sequence has a constant rate of change while the rate of change of the geometric sequence either increases or decreases. It is this connection that accounts for the computational power and efficiency the logarithm provides and justifies the importance of this discovery in the historical account of mathematical development. The map situates the conceptual genesis of a new mathematical idea, a realization of a connection between arithmetic and geometry. This relationship is the substantive understanding that requires mediational attention if students are to make sense of their work with logarithms. The mapping across two previously considered disjoint branches of mathematical thought is the core of the logarithm concept, an idea that became of prime interest with the later discovery of exponents and the emergence of calculus.

3.3 More Fully Developed Concept Map of a Logarithm

As is often the case, theoretical development leads to new technology. The new technology is applied to practical matters and simultaneously enables related theoretical development. The logarithm gave scientists a new conceptualization that proved immensely valuable in their effort to describe properties of the physical world using the language of mathematics. It should be noted here that the theoretical explanation found in the *Mirifici*, though completely consistent with calculus, was written prior to the existence of calculus. Points P and Q, moving along at different rates in figure 1, PB decreasing in geometric progression while CQ increases in arithmetic progression, would necessarily fall in the jurisdiction of the mathematics of change, namely calculus. Napier's geometric representation defined logarithms in kinetic terms and foreshadowed their significance in the development of calculus. Therefore it is not surprising that we find logarithms along side other transcendental functions in every modern calculus text. What you do not find in the *Mirifici* is any mention of logarithms as exponents. Bernoulli and others recognized this connection toward the end of the 17th century. "One of the anomalies in the history of mathematics is that logarithms were discovered before exponents were in use." (Eves, 1969)

In a sense, Figure 3 represents all that was known about logarithms in 1619. Continued mathematical conceptualization provides a concept map of far more extension and depth. The growth of relationships that develop in a scientific discipline rapidly create a complex structure that may mask the purity of the concept at the core.

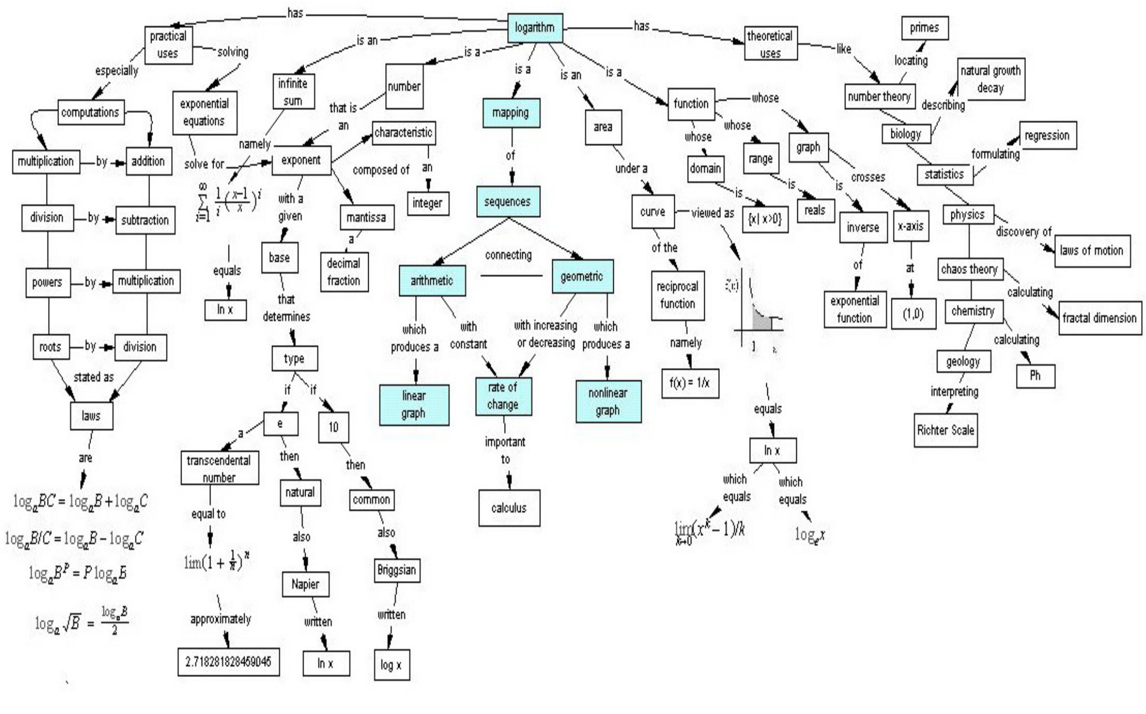

Figure 4. A concept map of logarithm with a cultural historical perspective as its central structure.

Figure 4 incorporates the cultural historical perspective as the central structure of a more detailed map of the logarithm concept. Generally, concepts arranged on the left are more strongly associated with the arithmetic notions and those on the right more geometric in nature. Practical uses appear on the left and theoretical connections on the right. It is interesting to note that the exponent emerges with such prominence in the map even though the concept was unknown when the logarithm relationship was first recognized. The same is true for the calculus related content. It too is well represented in the map though it had not yet been developed. This suggests the importance of cultural historical considerations in curriculum development. The concept exists without reference to these later developments.

The map in Figure 4 clearly reveals the extent of schema development that can arise from a simple consideration of points moving on a line. For the mathematics educator, the map complexity can be problematic if the essence of the concept, represented by the central structure, is missing. Instructionally, the concept loses its original meaning. Practical uses become independent procedures involving symbol manipulation. Theoretical connection becomes impossible, for there is no meaningful cross over access without substantive knowledge. Without conscious conceptual understanding of the essence of a scientific idea, mindful use is impossible.

4 The Problem of Generative Metonymy

Schmittau (2003) has identified the problem of generative metonymy as an impediment to mathematical understanding. Epistemological uncertainty accounts for considerable confusion in mathematical thought and the difficulty of understanding more advanced mathematics. As linguists Lakoff and Johnson (1980, p.39) remind us, “metonymic concepts allow us to conceptualize one thing by means of its relation to something else.” This use of language applied to scientific concepts masks the ideational essence and effects how a learner organizes their thoughts around new ideas. Multiple iterations of metonymy can develop completely inadequate conceptualizations leading to a form of intellectual desiccation. For students in this study, the ability to reason mathematically and problem solve with logarithms was minimal. As a result of generative metonymy the concept had lost its genetic meaning.

4.1 Conceptual Representation – the Teachers’ View

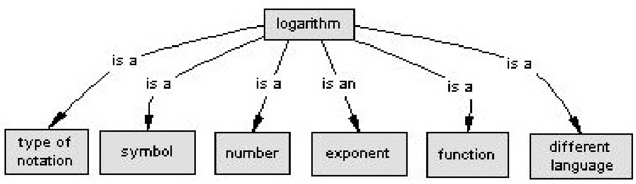

Figure 5. Composite Concept Map of Teachers Characterizations of a Logarithm

Teachers (Figure 5) variously describe a logarithm as: "a number," "a symbol," "a type of notation," "a function," "a different language," and "an exponent." This metonymy makes no mention of the essential characteristic, the mapping between geometric and arithmetic sequences. The primary understanding of teachers in this study is that a logarithm is an exponent. This is a central theme in their efforts to teach students this concept. The awkward and somewhat nebulous statement that a logarithm *is an exponent that you raise the base to to get a number* is often abbreviated to *a logarithm is an exponent.* The statement is highlighted, underlined, and made the key idea in logarithm chapters of each of the twelve textbooks investigated as part of this study. The statement doubly masks the essence of a logarithm and fails to provide the conceptual connection that would make sense of both practical and theoretical applications.

Of related importance are the views of teachers Fred, Maria and Steve. Fred teaches logarithms as a notational convenience. “They [the students] have a hard time with the definition because it's pure notation to them. It's a symbol, $\log_b x$.” Maria’s students are taught to solve exponential equations using logarithms, a process used as justification for the existence of the logarithmic concept. “The emphasis should be on what a logarithm is. I start with an equation that is impossible to solve without logs, something like $5^x = 112$.” Steve relies on the fact that a logarithm is a function. “I like to think of it as a function that returns a number. The log of x is going to give you back a number. What number? It's going to be the exponent, the log always returns an exponent.” These statements represent clear evidence of some teachers’ admitted algorithmic focus and the superficiality of understandings they present to students. There is little substance compared to the conceptual essence depicted in Figure 3.

4.2 Conceptual Representation – the Students’ View

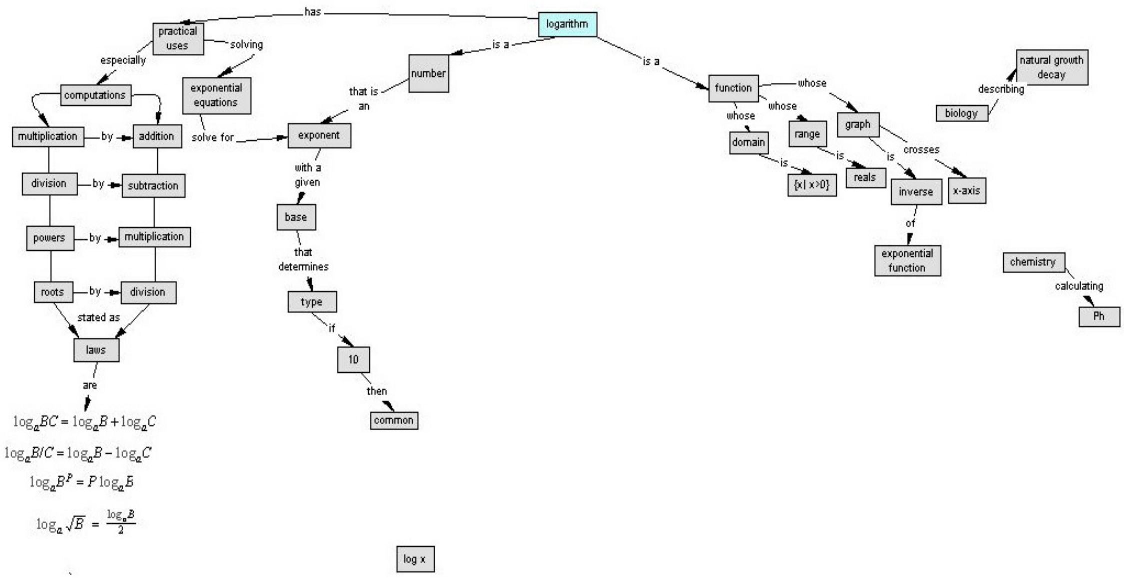

Figure 6. Concept map of logarithm from students’ perspective

The map in Figure 6 was created from the map in Figure 4 by removing anything not mentioned by students in six hours of interview. In this sense, the map in Figure 6 represents a composite of student understanding for the logarithm concept. Note that the critical core, the essence of a logarithm, is missing. The laws of logarithms and their use in solving exponential equations remain. Of the scientific applications seen in Figure 4, only pH and natural decay are cited. The student map reveals the limited nature of understanding and accounts for the inadequacy of mathematical reasoning expressed in interviews. How are students to understand the significance of reports of seismographic activity, for example? Will comparisons of 1.1 and 2.2 on the Richter scale be incorrectly interpreted as an earthquake of double intensity, when in fact the increase is tenfold due to the logarithmic nature of the scale? What sense will students make of Mandelbrot’s (1977) work on fractals when the formula to determine fractal dimensions depends on logarithms? How will students read with wonder, Bronofsky’s (1973) eloquent description of Ludwig Boltzmann’s formula S = K log W, entropy is in direct proportion to the logarithm of W, the probability of a given state. It was this formula that settled the theoretical debate over the existence of atoms and made possible current advancements in physics and biology. The student logarithm map revealed in Figure 6 is the direct result of generative metonymy, mediational inadequacy, and is void of conceptual essence. This lack of substantive knowledge makes mindful student use of logarithms unlikely.

Improved conceptual representation can positively impact student problem solving competency and improve their ability to apply their knowledge to related scientific work. It is possible to present the logarithm concept to students in a manner very similar to the geometric rendering of the definition offered by Napier. As Schmittau (1993, p.34) has stated, "pedagogical mediation must facilitate the appropriation of the scientific concept through a mode of presentation that reflects the objective content of the concept in its essential interrelationships." By doing so with logarithms, it may be possible to close the gap of conceptual understanding that is evident among students and similarly reveal the computational consequence that enabled the scientific community. As a result of this study, four mathematics lessons have been created toward that end. The first lesson addresses the essence and origin of the concept. The second lesson presents the link between the conceptualization of a logarithm as a relation between moving points and the graphical representation of that relation. The third lesson shows the impact of Napier's discovery on computational efficiency and accuracy. The final lesson investigates the use of logarithmic scales in mathematical reasoning. This curricular work in mathematics represents the positive contribution to be made by using concept mapping in conjunction with cultural historical research.

5 Summary

Vygotskian notions on cognitive development direct mathematics educators to identify the essence of concepts to be taught, reflecting with clarity the cultural historical context that produces new mathematical thought. Concept mapping, when combined with historical research, serves as an important epistemological tool that can, at once, render to consciousness the conceptual essence of a mathematical idea, identify for educators substantive focus for curriculum design, and provide pedagogical direction toward the mindful use of learned mathematics on the part of students. Meaningful instruction in mathematics avoids the temptation of a rote learning approach, recognizing that functional efficiency often leads to cognitive deficiency. Complex problem solving depends on the thoughtful application of meaningful mathematical ideas. Concept mapping can instruct and guide mathematics educators toward a pedagogy of significant cognitive consequence.

6 Acknowledgements

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References

Bronosky, J. (1973). *The ascent of man*. Boston: Little, Brown.

Cooke, R. (1997). *The History of Mathematics: A Brief Course.* New York: John Wiley & Sons.

Davydov, V. (1990). *Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula. Soviet studies in mathematics education* (Vol. 2). Reston, VA:National Council of Teachers of Mathematics.

Eves, H. (1969). *An introduction to the history of mathematics* (3rd. ed.). New York: Holt, Rinehart & Winston.

Lakoff, G. & Johnson, M. (1980). *Metaphors we live by*. Chicago, IL: University of Chicago Press.

Langer, E. (1989). *Mindfulness*. Reading, MA: Addison-Wesley.

Luria, A. R. (1976). *Cognitive development: Its cultural and social foundations.* (M. Lopez-Morillas & L. Solotaroff, trans.). Cambridge, MA: Harvard University Press.

Mandelbrot, B. (1977). *The fractal geometry of nature*. New York: W.H. Freeman.

Novak, J. & Gowan, D. (1984). *Learning how to learn*. Cambridge, UK: Cambridge University Press.

Schmittau, J. (1993).Vygotskian scientific concepts: Implications for mathematics education. *Focus on Learning Problems in Mathematics, 15* (2&3), 29-39.

Schmittau, J. (2003). Cultural-Historical Theory and Mathematics Education. In Kozulin A. & others (Eds.), *Vygotsky's Educational Theory in Cultural Context* (pp. 225-245). Cambridge University Press.

Srinivasiengar, C.N. (1967). *The History of Ancient Indian Mathematics.* Calcutta, World Press.

Turnbull, H. W. (1969). *The Great Mathematicians.* New York: New York University Press.

Vygotsky, L. (1978). *Mind in society*: *The development of higher psychological processes.* (M. Cole, V. John-Steiner, S. Scribner, E. Souberman, Eds.) Cambridge, MA: Harvard University Press.